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Dominance Theory and Plane Partitions

V. Enumeration of Ordinary Plane Partitions

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Let $E_{\lambda}^{(n)}(\bar{n};q)$ denote the GF for ordinary plane partitions of shape λ with part restrictions $\bar{n}$ and n (i.e., the parts in ith row are not less than n_i and the largest part in partitions does not exceed n). By establishing the correspondence between reverse plane partitions and ordinary plane partitions, we have derived the following determinant expression.

Theorem 5.1.

$$E_{\lambda}^{(n)}(\bar{n};q)=q^{\langle\bar{n},\bar{\lambda}\rangle}\det_{k\times k}\left\| \begin{bmatrix} n-n_i+\lambda_i \\ j-i+\lambda_i \end{bmatrix} q^{\binom{j-i}{2}+(j-i)n_i} \right\|$$

It has the following consequences.

Theorem 5.2.

$$E_{\lambda}^{(n)}(\bar{J};q)=q^{|\bar{\lambda}|+2n(\lambda)}\prod_{(i,j)\in\lambda}\frac{\langle n+c_{ij}\rangle}{\langle h_{ij}\rangle}$$

Theorem 5.3.

$$E_{\lambda}^{(n)}(\bar{\lambda};q)=q^{|\bar{\lambda}|+2n(\lambda')}\prod_{(i,j)\in\lambda}\frac{\langle n-c_{ij}\rangle}{\langle h_{ij}\rangle}$$

Corollary 5.4. (**MacMahon, 1915/1916**).

$$E_{r*c}^{(n)}(0;q)=\prod_{i=1}^{r}\begin{bmatrix} n+c+i-1 \\ c \end{bmatrix}\Big/\begin{bmatrix} c+i-1 \\ c \end{bmatrix}$$

And dually

$$E_{r*c}^{(n)}(0;q)=\prod_{j=1}^{c}\begin{bmatrix} n+r+j-1 \\ r \end{bmatrix}\Big/\begin{bmatrix} r+j-1 \\ r \end{bmatrix}$$

Corollary 5.5. (**MacMahon, 1915/1916**).

i. $E_{r*c}^{(\infty)}(0;q)=\prod_{i=1}^{r}\prod_{j=1}^{c}\langle i+j-1\rangle^{-1}$.

ii. $E_{r*\infty}^{(\infty)}(0;q)=\prod_{k=1}^{\infty}\langle k\rangle^{-\min(k,r)}$.

iii. $E_{\infty*\infty}^{(\infty)}(0;q)=\prod_{k=1}^{\infty}\langle k\rangle^{-k}$.

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