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积图$P_m\times C_{4n}$的k-优美性*

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对于正整数k,简单图$G=(V, E)$称为是k-优美的,若存在单一映射

$$f: V(G)\to\{0,1,2,\cdots,|E|+k-1\},$$

使得由之导出的映射

$$f^*: E(G)\to\{k, k+1, \cdots |E|+k-1\}$$
$$uv\to|f(u)-f(v)|$$

是一一映射.(参见〔1,2〕)

简单图$G_1=(V_1, E_1)$与$G_2=(V_2, E_2)$的积图$G=G_1\times G_2=(V, E)$指的是:$V=V_1\times V_2$,而点(v_1, v_2)与(v_1', v_2')间有边$\in E\Longleftrightarrow(v_1=v_1'$且$v_2v_2'\in E_2)$或$(v_2=v_2'$且$v_1v_1'\in E_1)$.

本文讨论积图$P_m\times C_{4n}$的k-优美性,这里m, n, k皆为正整数,而P_m表示m个点的链,C_{4n}表示$4n$个点的简单回路.

设

$$V(P_m)=\{T_0, T_1, \cdots, T_{m-1}\},$$
$$E(P_m)=\{T_iT_{i-1};\ 0\leqslant i\leqslant m-2\},$$
$$V(C_{4n})=\{S_0, S_1, \cdots, S_{4n-1}\},$$
$$E(C_{4n})=\{S_jS_{j-1};\ 0\leqslant j\leqslant 4n-2\}\cup\{S_{4n-1}S_0\},$$

而积图$P_m\times C_{4n}$(可视为m行$4n$列的格盘,但每行的两端点需连上一边)的点集和边集记为:

$$V(P_m\times C_{4n})=\{A_{ij};\ A_{ij}=(T_i, S_j), 0\leqslant i\leqslant m-1, 0\leqslant j\leqslant 4n-1\},$$
$$E(P_m\times C_{4n})=\{A_{i,j}A_{i,j-1};0\leqslant i\leqslant m-1,0\leqslant j\leqslant 4n-2\}\cup\{A_{i,4n-1}A_{i,0};$$
$$0\leqslant i\leqslant m-1\}\cup\{A_{i,j}A_{i-1,j};\ 0\leqslant i\leqslant m-2, 0\leqslant j\leqslant 4n-1\}. \qquad (**)$$

显然,$P_m\times C_{4n}$的边数$M'=4n(2m-1)$.记$M=M'+k-1$.

今给出积图$P_m\times C_{4n}$的顶点集的如下标号:

$$f(A_{i,j})=\begin{cases}4ni+\left[\dfrac{j+1}{2}\right]-\varepsilon_j & (i+j\overset{2}{\equiv}0\text{ 时})\\ M-4ni-\left[\dfrac{j}{2}\right]-\delta_j & (i+j\overset{2}{\equiv}1\text{ 时}),\end{cases}$$

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其中 $\varepsilon_j=\begin{cases}2n & (i\text{奇且}j=4n-1\text{时}),\\ 0 & (\text{否则})\end{cases}$ $\delta_j=\begin{cases}0 & (0\leqslant j\leqslant 2n-1\text{时})\\ 1 & (2n\leqslant j\leqslant 2n-1\text{时})\end{cases}$,

而对实数a，记号〔a〕表示不超过a的最大整数，我们并将$i\equiv 0,1(\mathrm{mod}2)$ 简记为$i\overset{2}{\equiv}0,1$. 以下证明所定义的f确是积图$P_m\times C_{4n}$的k-优美标号．分两大步进行．

step 1．f是$V(P_m\times C_{4n})\to\{0,1,\cdots,M\}$ 的单射．

先考察$i+j\overset{2}{\equiv}0$ 的点A_{ij}的标号值$f(A_{ij})$，易知，第 i 行中这样的点的f值两两不同，且恰充满正整数段〔$4ni,4ni+2n-1$〕；进而可知不同行的这些整数段两两不交（间隔距离$2n+1$)，且$P_m\times C_{4n}$全图中这种点的f值最大者为$4n(m-1)+2n-1=4mn-2n-1$，最小者为0.

再考察$i+j\overset{2}{\equiv}1$ 的点A_{ij}的标号值．不难看出，第 i 行中这样的点的f值亦两两不同，且恰充满两个正整数段的并:〔$M-(4i+2)n$, $M-(4i+1)n-1$〕$\cup$〔$M-(4i+1)n+1,M-4ni$〕; 进而可知不同行的这些整数段组也两两不交（间隔距离$2n$)，且图中这种点的f值最小者为$M-(4(m-1)+2)n=M-4(m-1)n-2n=4mn-2n+k-1$，最大者为$M$.

综上，因$k\geqslant 1$，故有

$$0<4mn-2n-1<4mn-2n+k-1<M.$$

亦即全部点的f值确落于集合$\{0,1,\cdots,M\}$中，且它们的f值两两不同．

step 2．f^*是$E(P_m\times C_{4n})\to\{k,k+1,\cdots,M\}$ 的双射．

这里的f^*当然是由所给的f按照k-优美的定义诱导出来的．为叙述简洁，对$E(P_m\times C_{4n})$的三类边（参见（**）式)，分别记其f^*值为:

$$b_{i,j}=f^*(A_{i,j}A_{i,j-1}),\quad 0\leqslant i\leqslant m-1,\quad 0\leqslant j\leqslant 4n-2\ ;$$
$$b_{i,4n-1}=f^*(A_{i,4n-1}A_{i,0}),\quad 0\leqslant i\leqslant m-1\ ;$$
$$C_{i,j}=f^*(A_{i,j}A_{i-1,j}),\quad 0\leqslant i\leqslant m-2,\quad 0\leqslant j\leqslant 4n-1\ .$$

我们并以$x\succ y$表示正整数x, y间满足$x=y+1$的关系．在证明中，还将用到以下事实:

① $\left[\frac{j}{2}\right]+\left[\frac{j+1}{2}\right]=j$;

② $\left[\frac{j+1}{2}\right]+\left[\frac{j+1}{2}\right]=\begin{cases}j & (j\overset{2}{\equiv}0\text{ 时})\\ j+1 & (j\overset{2}{\equiv}1\text{ 时})\end{cases}$;

③ $\left[\frac{j}{2}\right]+\left[\frac{j+2}{2}\right]=\begin{cases}j+1 & (j\overset{2}{\equiv}0\text{ 时})\\ j & (j\overset{2}{\equiv}1\text{ 时})\end{cases}$;

④ 当$j\in$〔0，$4n-2$〕且$j\overset{2}{\equiv}0$ 时$\delta_{j-1}=\delta_j$;

⑤ 当$j\in$〔0，$2n-2$〕$\cup$〔$2n$，$4n-2$〕且$j\overset{2}{\equiv}1$ 时$\delta_{j-1}=\delta_j$.

以下分五点来说明step 2 的结论．

1°．$C_{i,0}\succ C_{i,1}\succ\cdots\succ C_{i,2n-1}\overset{\triangle}{\succ}C_{i,4n-1}\overset{\triangle}{\succ}C_{i,2n}\succ C_{i,2n-1}\succ\cdots\succ C_{i,4n-2}$ $(0\leqslant i\leqslant m-2)$.

事实上，我们有$(0\leqslant i\leqslant m-2,\ 0\leqslant j\leqslant 4n-2)$:

$$C_{i,j}=\begin{cases}f(A_{i+1,j})-f(A_{i,j})=(M-4n(i+1)-\left[\frac{j}{2}\right]-\delta_j)-(4ni+\left[\frac{j+1}{2}\right]) & (i+j\overset{2}{\equiv}0\text{ 时})\\ f(A_{i,j})-f(A_{i+1,j})=(M-4ni-\left[\frac{j}{2}\right]-\delta_j)-(4n(i+1)+\left[\frac{j+1}{2}\right]) & (i+j\overset{2}{\equiv}1\text{时})\end{cases}$$

据①，两种情况下均有$C_{i,j}=M-4n(2i+1)-j-\delta_j$．注意到$\delta_j$的变化情况立知除去两个Δ外的全部关系，至于这两个Δ，可由下计算得知：

$$C_{i,2n-1}=M-4n(2i+1)-(2n-1)=M-2n(4i+3)+1\ ;$$

$$C_{i,4n-1}=\begin{cases} f(A_{i,4n-1})-f(A_{i+1,4n-1})=(M-4ni-\left[\frac{4n-1}{2}\right]-1)-(4n(i+1)+\left[\frac{4n}{2}\right]-2n) & (i\overset{2}{\equiv}0\text{时}) \\ f(A_{i+1,4n-1})-f(A_{i,4n-1})=(M-4n(i+1)-\left[\frac{4n-1}{2}\right]-1)-(4ni+\left[\frac{4n}{2}\right]-2n) & (i\overset{2}{\equiv}1\text{时}) \end{cases}$$

$$=M-2n(4i+3)\ ;$$

$$C_{i,2n}=M-4n(2i+1)-2n-1=M-2n(4i+3)-1\ .$$

2°．$b_{i,0}\succ b_{i,1}\succ\cdots\succ b_{i,2n-1}\overset{\triangle}{\succeq} b_{i,4n-1}\overset{\triangle}{\succeq} b_{i,2n}\succ b_{i,2n+1}\succ\cdots\succ b_{i,4n-2}$ $(0\leqslant i\leqslant m-1,\ i\overset{2}{\equiv}0)$．

事实上，我们有（$0\leqslant i\leqslant m-1,\ i\overset{2}{\equiv}0,\ 0\leqslant j\leqslant 4n-2$）：

$$b_{i,j}=\begin{cases} f(A_{i,j+1})-f(A_{i,j})=(M-4ni-\left[\frac{j+1}{2}\right]-\delta_{j+1})-(4ni+\left[\frac{j+1}{2}\right]) & (j\overset{2}{\equiv}0\text{时}) \\ f(A_{i,j})-f(A_{i,j+1})=(M-4ni-\left[\frac{j}{2}\right]-\delta_j)-(4ni+\left[\frac{j+2}{2}\right]) & (j\overset{2}{\equiv}1\text{时}) \end{cases}$$

据②，③，④，两种情况下均有$b_{i,j}=M-8ni-j-\delta_j$．注意到δ_j的变化情况立知除去两个Δ外的全部关系，至于所余情况，可由下知（$i\overset{2}{\equiv}0$）：

$$b_{i,2n-1}=M-8ni-(2n-1)=M-8ni-2n+1;$$

$$b_{i,4n-1}=f(A_{i,4n-1})-f(A_{i,0})=(M-4ni-\left[\frac{4n-1}{2}\right]-1)-(4ni+\left[\frac{1}{2}\right])$$

$$=M-8ni-2n;$$

$$b_{i,2n}=M-8ni-2n-1\ .$$

3°．$b_{i,4n-1}\overset{\triangle}{\succeq} b_{i,0}\succ b_{i,1}\succ\cdots\overset{\triangle}{\succeq} b_{i,2n-2}\overset{\triangle}{\succeq} b_{i,4n-2}\overset{\triangle}{\succeq} b_{i,2n-1}\succ b_{i,2n}\succ\cdots\succ b_{i,4n-3}$ $(0\leqslant i\leqslant m-1,\ i\overset{2}{\equiv}1)$．

事实上，我们有（$0\leqslant i\leqslant m-1,\ i\overset{2}{\equiv}1,\ 0\leqslant j\leqslant 4n-3,\ j\neq 2n-1$）

$$b_{i,j}=\begin{cases} f(A_{i,j})-f(A_{i,j+1})=(M-4ni-\left[\frac{j}{2}\right]-\delta_j)-(4ni+\left[\frac{j+2}{2}\right]) & (j\overset{2}{\equiv}0\text{ 时}) \\ f(A_{i,j+1})-f(A_{i,j})=(M-4ni-\left[\frac{j+1}{2}\right]-\delta_{j+1})-(4ni+\left[\frac{j+1}{2}\right]) & (j\overset{2}{\equiv}1\text{ 时}) \end{cases}$$

据②，③，⑤，两种情况下均有$b_{i,j}=M-8ni-j-1-\delta_j$．此即给出除去三个Δ外的全部关系，而所余情况可由以下计算知（$i\overset{2}{\equiv}1$）：

$$b_{i,4n-1}=f(A_{i,0})-f(A_{i,4n-1})=(M-4ni-\left[\frac{0}{2}\right])-(4ni+\left[\frac{4n}{2}\right]-2n)=M-8ni\ ;$$

$$b_{i,0}=M-8ni-0-1=M-8ni-1\ ;$$

$$b_{i,2n-2}=M-8ni-(2n-2)-1=M-8ni-2n+1\ ;$$

$$b_{i,4n-2}=f(A_{i,4n-2})-f(A_{i,4n-1})=(M-4ni-\left[\frac{4n-2}{2}\right]-1)-(4ni+\left[\frac{4n}{2}\right]-2n)$$

$$=M-8ni-2n\ ;$$

$$b_{i,2n-1}=f(A_{i,2n})-f(A_{i,2n-1})=\left(M-4ni-\left[\frac{2n}{2}\right]-1\right)-\left(4ni+\left[\frac{2n}{2}\right]\right)$$
$$=M-8ni-2n-1\ .$$

4°. $\begin{cases} b_{i,4n-2}\succ C_{i,0},\ C_{i,4n-2}\succ b_{i+1,4n-1} & (i\overset{2}{=}0,\ 0\leqslant i\leqslant m-1) \\ b_{i,4n-3}\succ C_{i,0},\ C_{i,4n-2}\succ b_{i+1,0} & (i\overset{2}{=}1,\ 0\leqslant i\leqslant m-1) \end{cases}$

事实上，由上述1°，2°，3°可直接得知

$$\left.\begin{aligned} &b_{i,4n-2}=M-8ni-(4n-2)-1=M-8ni-4n+1 \\ &C_{i,0}=M-4n(2i+1)-0=M-8ni-4n \\ &C_{i,4n-2}=M-4n(2i+1)-(4n-2)-1=M-8ni-8n+1 \\ &b_{i+1,4n-1}=M-8n(i+1)=M-8ni-8n \end{aligned}\right\}\ (i\overset{2}{=}0\text{ 时})$$

$$\left.\begin{aligned} &b_{i,4n-3}=M-8ni-(4n-3)-1-1=M-8ni-4n+1 \\ &C_{i,0}=M-8ni-4n \\ &C_{i,4n-2}=M-8ni-8n+1 \\ &b_{i+1,0}=M-8n(i+1)-0=M-8ni-8n \end{aligned}\right\}\ (i\overset{2}{=}1\text{ 时})$$

5°．综上1°—4°可知$P_m\times C_{4n}$的全部边的f^*值恰排成一个逐项严格减1的正整数列，诸f^*值中最大的一个是$b_{0,0}=M-8n\cdot 0-0=M$，最小的一个是

$$\begin{cases} b_{m-1,4n-2}=M-8n(m-1)-(4n-2)-1=k & (m\overset{2}{=}1\text{ 时}) \\ b_{m-1,4n-3}=M-8n(m-1)-(4n-3)-1-1=k & (m\overset{2}{=}0\text{ 时}) \end{cases}$$

这就完全证明了step 2．

附注．本文所讨论的积图$P_m\times C_{4n}$可直观地视为具有$m-2$个横截面的$4n$棱柱．(当$m\geqslant 2$时，而若$m=1$，则是一个$4n$边形)．我们的标号方法建基于C_{4n}(即$4n$边形）的如下“k-优美标号”途径：

$$f(A_j)=\begin{cases} \lambda+\dfrac{j}{2} & (j\overset{2}{=}0) \\ k+\lambda+4n-1-\dfrac{j-1}{2}-\delta_j & (j\overset{2}{=}1) \end{cases}$$

其中A_0，A_1，…，A_{4n-1}表示C_{4n}的全部顶点，λ为任意给定的非负整数，

$$\delta_j=\begin{cases} 0 & (1\leqslant j\leqslant 2n-1) \\ 1 & (2n+1\leqslant j\leqslant 4n-1) \end{cases}$$

此时的f是$V(C_{4n})\to\{\lambda,\lambda+1,\cdots,k+\lambda+4n-1\}$的单射，而$f^*$则将是$E(C_{4n})\to\{k,k+1,\cdots,k+4n-1\}$的双射．

不难看出，对C_n顶点集的任一标号方法f，都必有其全部边的f^*值的和为偶数（事实上，若设各顶点的f值依次为a_0，$a_1,\cdots,a_{n-1}$，则全部边的f^*值的和为$\sum\limits_{i=0}^{n-2}|a_i-a_{i+1}|+|a_{n-1}-a_0|\overset{2}{=}\sum\limits_{i=0}^{n-2}(a_i-a_{i+1})+a_{n-1}-a_0=0$）而由$k$-优美的要求，需$\sum\limits_{i=0}^{n-1}(k+i)=nk+\dfrac{n(n-1)}{2}\overset{2}{=}0$．显然这对$n\overset{4}{=}2$是不可能的；而对$n\overset{4}{=}1$，需$k$偶；对$n\overset{4}{=}3$，需$k$奇；只有对$n\overset{4}{=}0$才允许$k$任意．因此本文之方法仅对$n\overset{4}{=}0$进行了讨论．当然，对$n\overset{4}{\neq}0$的情况，给出$P_m\times C_n$的$k$-优美标号，

并不排斥由其它途径（即不再是每横行的边值可以自成一个连续整数段）给出的可能，这是值得进一步研究的。

参 考 文 献

〔1〕P.J.Slater, On k-graceful graphs, Proc of the 13th S. E. Conference on Combinatorics, Graph Theory and Computing, Boca Raton, 1982, 52—57.

〔2〕M. Maheo and H. Tkuillier, On d-graceful graphs, Arts Combinatoric, vol.13(1982), 181—192.

The k-gracefulness on the Product of graphs p_m and c_{4n}

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Abstract

Let k be a positive integer. The simple graph $G=(V, E)$ is called k-graceful if there exists a injection.

$$f: V(G) \to \{0, 1, \cdots, |E|+k-1\}$$

such that following induced mapping is bijection:

$$f^*: E(G) \to \{k, k+1, \cdots, |E|+k-1\}$$

$$f^*(uv)=|f(u)-f(v)| \quad (u, v \in V(G), uv \in E(G)).$$

The product of the simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ is a graph $G=G_1\times G_2=(V, E)$, whose vertex set is $V=V_1\times V_2$ and whose edge set is consisted as follows:

vertices(v_1, v_2) and (v_1', v_2') is adjacent if and only if ($v_1=v_1'$ and $v_2v_2'\in E_2$)or ($v_2=v_2'$ and $v_1v_1'\in E_1$).

In this paper we verify the k-gracefulness on product graph $P_m\times C_{4n}$, where m, n and k are positive integers, p_m denotes a chain with m vertices and c_{4n} denotes a circle with $4n$ vertices.

Let the vertex set of the graph $p_m\times c_{4n}$ be

$$\{A_{i,j};\ 0\leqslant i\leqslant m-1,\ \ 0\leqslant j\leqslant 4n-1\}$$

and $M=4n(2m-1)+k-1$. The k-graceful Label an the graph $p_m\times c_{4n}$ given by us is:

$$f(A_{i,j})=\begin{cases} 4ni+\left[\dfrac{j+1}{2}\right]-\varepsilon_{ij} & (i+j\equiv 0 \bmod 2) \\ M-4ni-\left[\dfrac{j}{2}\right]-\delta_j & (i+j\equiv 1 \bmod 2) \end{cases}$$

where $\varepsilon_{ij}=\begin{cases} 2n & (j=2n-1 \text{ and } i\equiv 1 \bmod 2) \\ 0 & (\text{else}) \end{cases}$ and $\delta_j=\begin{cases} 0 & (0\leqslant j\leqslant 2n-1) \\ 1 & (2n\leqslant j\leqslant 4n-1) \end{cases}$.