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Solvability of Finite Groups which Admit a Fixed-Point-Free Automorphism Group*

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The following conjecture is well known. "Let G be a finite group and $A\leqslant \mathrm{Aut}(G)$. If $(|G|, |A|)=1$ and $C_G(A)=1$, then G is solvable". As a special case, we pay much more attention to the case of that A is a p-group. When A is a p-group, the hypothesis "$(|G|,|A|)=1$" can be derived from "$C_G(A)=1$" by use of orbit formula. So far as we know, the possitive answers were given to the following cases: (1). $|A|=p$, (Thompson's theorem [6]); (2). $|A|=p^2$, ([1] Theorem 10 4.2, 10.5.6 and [4]); (3). A is an elementary abelian p-group, (Martineau's theorem [2]). But there is no general answer yet if the power index is larger than 2. In this paper, we give an elementary proof to the case of $p=2$ or 3 by use of a Glauberman's result.

Glauberman's theorem[5] Let G be a finite nonabelian simple group. If G is S_4 free, then G is isomorphic to one of the following simple groups:

(a), $S_z(2^r)$, r a prime; (b), $\mathrm{psl}(2,2^r)$, $r>1$; (c), J_1 or Ree; (d), $\mathrm{psl}(3,2)$; (e), $\mathrm{psl}(2,q)$, $q=3$ or $5 \pmod 8$.

Thompson's theorem[6] If a finite group G admits a fixed-point-free automorphism of prime order, then G is nilpotent.

Main theorem Let G be a finite group and $A\leqslant\mathrm{Aut}(G)$. If $12\nmid|G|$, $(|G|, |A|)=1$ and $C_G(A)=1$, then G is solvable.

Proof Suppose that this is false and consider a counterexample for which $|G|+|A|$ is minimal.

(1). Every proper A-invariant subgrup of G and every proper A-invariant quotient group of G is solvable.

Let G_1 be a proper A-invariant subgroup of G. Then A induces a group B_1 of automorphisms of G_1 and $|G_1|+|B_1|<|G|+|A|$, $C_{G_1}(B_1)=1$ and $12\nmid|G_1|$, $(|G_1|, |B_1|)=1$. Hence G_1 is solvable by the choice of G. Let N be a non-identity A-

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invariant normal subgroup of G, since $(|G|, |A|)=1$, we have that $C_{G/N}(A)=C_G(A)N/N=1$ by [2]X.11.1. Hence G/N is solvable by the same argument.

(2). G has no non-identity proper A-invariant normal sugroup. $G=G_1\times G_2\times\cdots\times G_m$ is a direct product of isomorphic non-abelian simple groups, where each G_i is a minimal normal subgroup of G.

Suppose that this is false and let $1\neq N\neq G$ and N be a normal A-invariant subgroup of G. By (1), both N and G/N are solvable and so is G. This contradiction implies that G is a characteristic simple group.

(3). G is a nonabelian simple group.

By (2), we only need to show that $m=1$. If $m>1$, Let $B=N_A(G_1)$ then $B<A$ by (1). Let $A=Ba_1+Ba_2+\cdots+Ba_n$ be the docomposation of all cosets of B in A, where $1=a_1, a_2, \cdots, a_n$ are the representive elements of cosets. Since G_1 is a minimal normal subgroup of G, so is $G_1^{a_i}$. $G_1^{a_i}\cap G_1^{a_j}\neq 1$ if and only if $G_1^{a_i}=G_1^{a_j}$ if and only if $a_i=a_j$. It is easy to show that $G_1^{a_1}\times G_1^{a_2}\times\cdots\times G_1^{a_n}$ is a non-identity A-invariant normal subgroup of G and so that $G=G_1^{a_1}\times G_1^{a_2}\times\cdots\times G_1^{a_n}$ by (2). Let $B_1=B/C_B(G_1)\leqslant \mathrm{Aut}(G_1)$. We have $|G_1|+|B_1|<|G|+|A|$. Since G_1 is nonsolvable, $C_{G_1}(B_1)=C_{G_1}(B)\neq 1$ by the choice of G. There exists $1\neq g_1\in C_{G_1}(B)$. Note that $G=G_1^{a_1}\times G_1^{a_2}\times\cdots\times G_1^{a_n}$. Let $g=g_1^{a_1}\cdots g_1^{a_n}$. It is easy to show that $1\neq g\in C_G(A)=1$. This is a contradiction.

Since $12\nmid|G|$, G is S_4 free. By (3), G is isomorphic to one of the simple groups listed in Glauberman's theorem. By [1] Theorem 7.6.1 and odd order theorem, $4\,\big|\,|G|$, hence $3\nmid|G|$. But $3\,\big|\,|G|$ if G is of the form (b) to (e) listed in Glauberman's theorem by [1] p490, hence $G=S_z(2^r)$, where r is a prime. By [3], $|\mathrm{Aut}(S_z(2^r))|=|S_z(2^r)|\cdot r$. Since $(|G|, |A|)=1$ and $A\neq 1$, we have $|A|=r$. It follows that G is nilpotent by Thompson's theorem.

The last contradiction completes our proof.

Corollary Let G be a finite group. Let $A\leqslant\mathrm{Aut}(G)$ be a p-group with $/p=2$ or 3. If $C_G(A)=1$, then G is solvable.

Proof Consider A acts on G. By orbit formula, we have $|G|=|M_1|+|M_2|+\cdots+|M_k|$, where $M_1, M_2, \cdots, M_k$ are the orbits under the action with $M_1=1$. Since $|M|\,\big|\,|A|=p^l$, $C_G(A)=1$, we have $|M|>1$ for $i=2,3,\cdots,k$ and $|G|\equiv 1 \pmod p$. Since $p=2$ or 3, $12\nmid|G|$. The Main theorem follows that G is solvable.

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关于允许一个无不动点自同构群的有限群的可解性

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摘要 关于允许一个无不动点自同构（群）的有限群的可解性的猜想是有限群研究中的一个重要问题．结果比较丰富的是限制该自同构群为一个 p-群的情形．Thompson 于1959年证明了 p 阶群的情形．Martineail 于1971年证明了初等 Abel p-群的情形．Rickman 于1979年证明了 p^2 阶群的情形．本文借助Glauberman的一个定理，对 $p=2$或3的一般情形给出了肯定的回答．实际上是用较初等的方法证明了更为广泛一些的结论．