Journal of Mathematical Research and Exposition
Vol.10 No.1 Feb. 1990

On the Joint Essential Spectrum, the Joint Essential Numerical Range and the Boundary of Joint Numerical Range

Cao Guangfu

(Harbin Architectural and Civil Engineering Institute)

Abstract The present note gives some results of the joint essential spectrum, the joint essential numerical range and the boundary of the joint numerical range for several commuting operators.

Ⅰ. Joint essential spectrum and joint essential numerical range

Definition Ⅰ.1 (R.E.Curto [3]) Let $T=(T_1\cdots T_n)$ be an n-tuple of almost commuting operators on Hilbert space H(i.e. $T_iT_j=T_iT_j-T_jT_i\in\mathscr{K}(H)$, $i,j=1,2,\cdots n$), and the Kozul complex

$$0\to\cdots\to E^n_{k+1}(H,C)\xrightarrow{d^{k+1}_n}E^n_k(H,C)\xrightarrow{d^k_n}E^n_{k-1}(H,C)\to\cdots\to 0 \tag{1.1}$$

is obtained from the system

$$0\to\cdots\to E^n_{k+1}(H,C)\xrightarrow{D^{k+1}_n}E^n_k(H,C)\xrightarrow{D^k_n}E^n_{k-1}(H,C)\to\cdots\to 0 \tag{1.2}$$

by the classical map. Where

$$D^k_n(xe_{j1}\wedge\cdots\wedge e_{jk})=\sum_{i=1}^{k}(-1)^{i+1}T_{ji}xe_{j1}\wedge\cdots\wedge e_{ji}\wedge\cdots\wedge e_{jk}$$

If the complex (1.1) is exact, then we say that T is a Fredholm tuple.

Definition Ⅰ.2 Let $T=(T_1\cdots T_n)$ be an n-tuple of almost commuting operators on infinite dimmensional Hilbert space H. If $\lambda\in C^n$ such that $T-\lambda$ is not a Fredholm tuple, then we say λ belongs to the joint essential spectrum of T. The joint essential spectrum of T is writed by $\mathrm{Sp}_e(T,H)$.

Definition Ⅰ.3 Let $T=(T_1\cdots T_n)$ be a n-tuple of several bounded linear operators on Hilbert space H. The set

$$w_{ess}(T)=\{\lambda\in C^n\,|\,\text{there is a sequence }\{x_k\}\subset H,\ \|x_k\|=1,$$
$$x_k\xrightarrow{w}0,\ \text{such that }(Tx_k,\ x_k)\to\lambda\}$$

is called the joint essential numerical range of T.

Theorem Ⅰ.4 Let $T=(T_1\cdots T_n)$ be an n-tuple of almost commuting operators on infinite dimmensional Hilbert space H, and $T_iT_j^*=T_j^*T_i$ $(i\neq j)$. Then $\mathrm{Sp}_e(T,H)\subset w_{ess}(T)$.

Proof Let $\lambda=(\lambda_1\cdots\lambda_n)\in\mathrm{Sp}_e(T,H)$, i.e. $T-\lambda$ isn't a Fredholm tuple. Then

*Received Jun.22, 1988.

there is a map f: $\{1,2,\cdots,n\}\to\{0,1\}$ such that $\sum_{i=1}^{n}{}^{f}(T_i-\lambda_i)$ isn't a Fredholm operator by Cor.3.7 in R.E.Curto [3]. Since $\sum_{i=1}^{n}{}^{f}(T_i-\lambda_i)$ is a positive operator. So $\sigma_e^l(\sum{}^f(T_i-\lambda_i))=\sigma_e(\sum{}^f(T_i-\lambda_i))$. Hence, for any $S\in L(H)$, $S(\sum{}^f(T_i-\lambda_i))-I$ is not a compact operator. By theorem 1.1 in P.A.Fillmore, J.G.Stampfli and J.P.Williams [2], there is a sequence $\{x_k\}\subset H$, $\|x_k\|=1, x_k\xrightarrow{w}0$, such that

$$\|(\sum_{i=1}^{n}{}^{f}(T_i-\lambda_i))x_k\|\to 0 \quad (k\to\infty)$$

consequently $\|{}^f(T_i-\lambda_i)x_k\|\to 0$ $(k\to\infty)$. Hence

$$\|(T_i-\lambda_i)x_k\|\to 0 \text{ or } \|(T_i-\lambda_i)^*{}_k\|\to 0.$$

Furthermore $((T_i-\lambda_i)x_k, x_k)\to 0$ (as $k\to\infty$), i.e. $\lambda=(\lambda_1,\cdots,\lambda_n)\in w_{ess}(T)$.

Theorem 1.5 Let $T=(T_1\cdots T_n)$ be an n-tuple of almost commuting essential operators on Hilbert space H. Then $\mathrm{Sp}_e(T,H)\subset W_{ess}(T)$.

Proof It is an immediate consequence of Cor. 3.9 in R.E.Curto [3].

Lemma 1.6 (G.F.Cao [1]) Let $T=(T_1\cdots T_n)$ be an n-tuple of bounded linear operators on Hilbert space H. $X=\bigoplus_{i=1}^{2^{n-1}}x_i\in\bigoplus_{1}^{2^{n-1}}H$, $(T_1\cdots T_n)_j^\wedge$ and $(T_1\cdots T_n)_j^{\wedge *}$ are respective the j-th rows of $(T_1\cdots T_n)^\wedge$ and $(T_1\cdots T_n)^{\wedge *}$.

$$J_{ji}(X,n)=((T_1\cdots T_n)_j^\wedge X, x_i),\quad J_{ji}^*=((T_1\cdots T_n)_j^{\wedge *}X, x_i).$$

Then we have followings

$$\sum_{i=1}^{2^{n-1}}(T_l x_i, x_i)=\sum_{i,j=1}^{2^{n-1}}a_{ij}^l J_{ji}(X,n)+\sum_{i,j=1}^{2^{n-1}}b_{ij}^l\overline{J_{ji}(X,n)} \tag{1}$$

$$\sum_{i=1}^{2^{n-1}}(T_l^* x_i, x_i)=\sum_{i,j=1}^{2^{n-1}}a_{ij}^l J_{ij}^*(X,n)+\sum_{i,j=1}^{2^{n-1}}b_{ij}^l\overline{J_{ij}^*(X,n)}, \tag{2}$$

where $a_{ij}^l, b_{ij}^l\in Z$ $(l=1,2,\cdots n)$.

Theorem 1.7 Let $T=(T_1\cdots T_n)$ be an n-tuple of almost commuting operators on infinite dimmensional Hilbert space H. Then $\mathrm{Sp}_e(T,H)\subset \mathrm{con}W_{ess}(T)$.

Proof Suppose $\lambda=(\lambda_1;\cdots,\lambda_n)\in\mathrm{Sp}_e(T,H)$, then $(T-\lambda)^\wedge=(T_1-\lambda_1,\cdots,T_n-\lambda_n)^\wedge$ is not a Fedholm operator by Cor.3.11 in R.E.Curto [3], i.e. $(\widetilde{T-\lambda})=(((T_1-\lambda_1 T_2-\lambda_2)$ diag $(T_3-\lambda_3)\cdots$ diag $(T_n-\lambda_n))^\wedge$ is not a Fredholm operator. So for any $S\in L(\bigoplus_{1}^{2^{n-1}}H)$, $S(\widetilde{T-\lambda})-I$ is a noncompact operator or $(T-\lambda)S-I$ is a noncompact operator. Hence there is a sence there is a sequence $\{\bigoplus_{i=1}^{2^{n-1}}x_i^m\}\subset\bigoplus_{1}^{2^{n-1}}H$, $\|\bigoplus_{i=1}^{2^{n-1}}x_i^m\|=1$, $\bigoplus_{i=1}^{2^{n-1}}x_i^m\xrightarrow{w}0$, such that $\|(T-\lambda)^\wedge(\bigoplus_{i=1}^{2^{n-1}}x_i^m)\|\to 0$ or $\|(T-\lambda)^{\wedge *}(\bigoplus_{i=1}^{2^{n-1}}x_i^m)\|\to 0$ by theorem 1.1 in P.A.Fillmore J.G.Stampfli and J.P.Williams [2]. Furtherm more $J_{ji}(\bigoplus_{l=1}^{2^{n-1}} x_l^m,\ n)\to 0$ or $J_{ji}^*(\bigoplus_{l=1}^{2^{n-1}} x_l^m, n)\to 0$. By lemma 1.6, we have

$$\sum_{l=1}^{2^{n-1}} ((T_i-\lambda_i)x_l^m,\ x_l^m)\to 0\ (i=1,2,\cdots n), \text{i.e.} \sum_{l=1}^{2^{n-1}} (T_i x_l^m,\ x_l^m)\to\lambda_i.$$

Without loss of generality, suppose $\|x_l^m\|^2\to\delta_l\,(m\to\infty)$, then $\sum_{l=1}^{2}\delta_l=1$. Let I_0 be the set of l such that $\delta_l\neq 0$, and suppose $\lim_m (T_i x_l^m/\|x_l^m\|,\ x_l^m/\|x_l^m\|)=\mu_i^l$, $l\in I_0$, $i=1,2,\cdots,\ n$. Then $\mu^l=(\mu_1^l\cdots\mu_n^l)\in W_{\text{ess}}(T)$, $l\in I_0$. Hence

$$\lambda=(\lambda_1\cdots\lambda_n)=\lim(\sum_{l=1}^{2^{n-1}}(T_1x_l^m,x_l^m)\cdots\sum_{l=1}^{2^{n-1}}(T_nx_l^m,x_l^m))$$
$$=\lim(\sum_{l\in I_0}\|x_l^m\|^2(Tx_l^m/\|x_l^m/\|x_l^m/\|x_l^m\|)=\sum_{l\in I_0}\delta_l\mu^l\in\text{con}W_{\text{ess}}(T).$$

2. Joint essential numerical range and boundary of joint numerical range

Proposition 2.1 Let $T=(T_1\cdots T_n)$ be an n-tuple of bounded linear operators on infinite dimmensional Hilbert space H, $K=(K_1\cdots K_n)$ be an n-tuple of compact operators. Then $W_{\text{ess}}(T)=W_{\text{ess}}(T+K)$. Where $T+K=(T_1+K_1\cdots T_n+K_n)$.

Proof Let $\lambda\in W_{\text{ess}}(T)$, then there is a sequence $\{x_k\}\subset H$, $\|x_k\|=1, x_k\xrightarrow{w}0$, such that $(T_ix_k,\ x_k)\to\lambda_i$, $\lambda=(\lambda_1\cdots\lambda_n)$. Suppose $K=(K_1\cdots K_n)$ be any n-tuple of compact operators, then $(K_ix_k,x_k)\to 0$. Hence $((T_i+K_i)x_k,x_k)\to 0$. Furthermore $\lambda\in W_{\text{ess}}(T+K)$. Similarly $W_{\text{ess}}(T+K)\subset W_{\text{ess}}(T)$.

Remark Clearly, we have $W_{\text{ess}}(T)\subseteq\bigcap_{\substack{K\in\mathcal{K}(H)\\K=(K_1\cdots K_n)}}\overline{W(T+K)}$.

where $\mathcal{K}(H)$ is the algebra of compact operators.

Proposition 2.2 $W_{\text{ess}}(T)$ be non-empty set.

Proof For any $\{x_k\}\subset H$, $\|x_k\|=1$, $(x_k,x_k')=0\ (k\neq k')$, we have $x_k\xrightarrow{w}0$. We can find a sub-sequence $\{x_{ki}\}_{i=1}^{\infty}$ such that $(T_ix_{kj},\ x_{kj})\to\lambda_i$. Hence $\lambda=(\lambda_1\cdots\lambda_n)\in W_{\text{ess}}(T)$.

Proposition 2.3 Let $T=(T_1\cdots T_n)$ be an n-tuple of commuting unilateral weighted shifts with no zero weights. Then $W_{\text{ess}}(T)$ is a convex set.

Proof It is easy. We omit it.

Proposition 2.4 Let $T=(T_1\cdots T_n)$ be an n-tuple of commuting operators on Hilbert space H, and for any two-dimmensional orthogonal projection P, we have $T_iPT_j=T_jPT_i$, $i,j=1,2,\cdots n$. Then $W_{\text{ess}}(T)$ is a convex set.

Proof Let $s=(s_1\cdots s_n)$, $\lambda=(\lambda_1\cdots\lambda_n)\in W_{\text{ess}}(T)$, $s\neq\lambda$. Then there are sequences $\{x_k\}$, $\{y_k\}\subset H$, such that $x_k\xrightarrow{w}0, y_k\xrightarrow{w}0$, $\|x_k\|=\|y_k\|=1$, and $(Tx_k,\ x_k)\to s$, $(Ty$ $(Tx_k,x_k)\to\lambda$. Let $H_k=V\{x_k,y_k\}$, then $H_k\subset H$ is two-dimmensional Hilbert space, By hypothesis of the proposition, $P_{xy}TP_{xy}=(P_{xy}T_1P_{xy}\cdots P_{xy}TP_{xy})$ is a n-

tupl of commuting operators, and $W(P_{xy}TP_{xy})=W(P_{xy}T|_{H_k})\subset W(T)$. Where P_{xy} is the projection onto H_k. Since $W(P_{xy}TP_{xy})$ is a convex set. For any $0\leqslant t\leqslant 1$, there is a $u_k\in H_k$, $\|u_k\|=1$, such that $(Tu_k, u_k)=t(Tx_k, x_k)+(1-t)\times(Ty_k, y_k)$. Suppose $u_k=\alpha_k x_k+\beta_k y_k$, we'll prove that both α_k and β_k are bounded, By $s\neq\lambda$, we have $|(x_k, y_k)|<\theta$, $\theta<1$. In fact, if $s\neq\lambda$, then there is a T_i such that $\lim\limits_{k=\infty}(T_ix_k, x_k)\neq\lim\limits_{k\to\infty}(T_iy_k, y_k)$. Suppose $y_k=\lambda_k x_k+\mu_k z_k$, $z_k\perp x_k$, then $(x_k, y_k)=\lambda_k$. If $\mu_k\to 0$, then $(y_k, y_k)=(|\lambda_k|^2(x_k, x_k)+|\mu_k|^2(z_k, z_k)=1$, so $|\lambda_k|=1$. It contradicts $s\neq\lambda$. Hence $\mu_k\to 0$. Furthermore $\lambda_k\to 1$, i.e. $|(x_k, y_k)|\leqslant\theta$, $\theta<1$. Without loss of generality, we assume $\alpha_k\to\alpha$, $\beta_k\to\beta$. Then $u_k\xrightarrow{w}0$, Hence $ts+(1-t)\lambda=\lim\limits_{k\to\infty}(Tu_k, u_k)\in W_{ess}(T)$.

Proposition 2.5 Let $T_k=(T_{1k}\cdots T_{nk})(k=1,2,\cdots m)$ be respective n-tuples of commuting operators on Hilbert spaces $H_k(k=1,2,\cdots m)$ and $W_{ess}(T_k)$ be all convex sets. Then $W_{ess}(\bigoplus\limits_{k=1}^{m}T_k)=\text{con}(\bigcup\limits_{k=1}^{m}W_{ess}(T_k))$.

Proof Suppose $\lambda=\sum\limits_{k=1}^{m}t_k\lambda_k\in\text{con}(\bigcup\limits_{k=1}^{m}W_{ess}(T_k))(\sum\limits_{k=1}^{m}t_k=1, t_k\geqslant 0)$, then there are sequences $\{x_{k_j}\}_{j=1}^{\infty}\subset H_k$, $\|x_{kj}\|=1, x_{k_j}\xrightarrow{w}0$, such that $(T_kx_{k_j}, x_{k_j})\to\lambda_k$. Let $x_j=\bigoplus\limits_{k=1}^{m}\sqrt{t_k}x_{k_j}$, then $x_j\xrightarrow{w}0$, $\|x_j\|=1$, and $(\bigoplus\limits_{k=1}^{m}T_kx_j, x_j)\to\sum\limits_{k=1}^{m}t_k\lambda_k$. Hence con $(\bigcup\limits_{k=1}^{m}W_{ess}(T_k)\subset W_{ess}(\bigoplus\limits_{k=1}^{m}T_k)$. Conversely, if $\lambda\in W_{ess}(\bigoplus\limits_{k=1}^{m}T_k)$, then there is a sequence $\bigoplus\limits_{k=1}^{m}x_{k_j}\subset\bigoplus\limits_{k=1}^{m}H_k$, $\sum\limits_{k=1}^{m}\|x_{k_j}\|^2=1, x_{k_j}\xrightarrow{w}0\ (j\to\infty)$, such that $((\bigoplus\limits_{k=1}^{m}T_k)(\bigoplus\limits_{k=1}^{m}x_{k_j}, \bigoplus\limits_{k=1}^{m}x_{k_j})=\sum\limits_{k=1}^{m}(T_kx_{kj}, x_{k_j})\to\lambda$. Hence $\lambda\in\text{con}(\bigcup\limits_{k=1}^{m}W_{ess}(T_k))$.

Theorem 2.6 Let $T=(T_1\cdots T_n)$ be an n-tuple of commuting operators on Hilbert space H. $W(T)$ is a convex set. Then $\text{Ext}\overline{W(T)}\subset W_{ess}(T)\ UW(T)$. In particular $\overline{W(T)}=\text{con}(W_{ess}(T)UW(T))$.

Proof For any $n\times n$ unitary matrix U and $\beta\in C^n$, we have $W(UT+\beta)=UW(T)+\beta$. One can assume $0\in\text{Ext}\ \overline{W(T)}-W(T)$, hence there is a sequence $\{x_k\}\subset H$, $\|x_k\|=1$, such that $(T_ix_k, x_k)\to 0$ $(i=1,2,\cdots n)$. Without loss of generality, we assume $x_k\xrightarrow{w}x$, $\|x\|\leqslant 1$, we claim $x=0$. In fact, if $\|x\|=1$, then $x_k\xrightarrow{s}x$, So $|(T_ix,x)|<|(T_i(x-x_k),x)|+|(T_ix_k, x-x_k)|+|(T_ix_k,x_k)|\to 0$, Hence $(T_ix,x)=0$, i.e. $0\in W(T)$. It contradicts our assumption. If $0<\|x\|<1$, then $(T_i(x-x_k), x-x_k), x-x_k)=(T_ix, x-x_k)-(T_ix_k, x)+(T_ix_k, x_k)\to-(T_ix, x)$, $\|x-x_k\|^2=1-2\text{Re}(x-x_k, x)-\|x\|^2$. Let $y_k=(x-x_k)/\|x-x_k\|$, then (T_iy_k, y_k)

$\to -(T_i x, x)/(1-\|x\|^2)$. However, this implies $0 \in \mathrm{Ext}\overline{W(T)}$. Thus completing the proof of the theorem.

We state following lemma2.7 – 2.14 but omit their proofs.

Lemma 2.7 Let G be a bounded non-empty closed set in C^n. Then there are at least two extreme points in G.

Definition 2.8 Let G be a convex sub-set in C^n, $\mathscr{C} = \{\Delta : \Delta \subset G$ is homeomorphic to a simplex$\}$, we say z is a k-th dimmensional interior point of G. If there is a $\Delta \in \mathscr{C}$ such that z lies in interior of Δ, and $\dim(\Delta) = k$, $k \leqslant 2n$. The set of k-th dimmensional interior points of G is writed by $\mathring{G}(k)$. Clearly, $\mathring{G}(k) \subset \mathring{G}(k-1)$.

Definition 2.9 Let G be a convex subset in C^n. The k th dimmensional boundary of G is the set $\partial_k G = G - \mathring{G}(k)$.

Lemma 2.10 Let G be a simply connected closed subset in C^n. $\mu \in G$, $\mathscr{M} = \{F(\mu) | F(\mu) \subset G$ is a closed convex subset, and $\mu \in F(\mu)\}$. Then there is a maximal element in $\mathscr{M}$. We denote the maximal element in $\mathscr{M}$ by $G(\mu)$.

Lemma 2.11 Let G be a bounded convex closed subset in C^n, and $\dim(G) = k$. If l is a ray that the center lies in $G(k)$, and there are at least two points in $l \cap G$. Then $l \cap \partial_k G$ is a simple point set.

Lemma 2.12 Let G be a bounded convex closed subset, and $\dim(G) = k$. Then $\mathrm{Ext}\partial_k G \subset \mathrm{Ext} G$.

Lemma 2.13 Let G be a bounded convex closed subset, and $\dim(G) = k$, $\mu \in \partial_k G$. Then $\mathrm{Ext}\partial_k G(\mu) \subset \mathrm{Ext}\partial_k G$.

Lemma 2.14 Let G be a bounded convex closed subset, $\mu \in G$. Then either μ is a extreme point of G or there are $\mu_1 \cdots \mu_m \in \mathrm{Ext} G$ such that $\mu \in \mathrm{con}\{\mu_1 \cdots \mu_m\}$.

Theorem 2.15 Let $T = (T_1 \cdots T_n)$ be an n-tuple of commuting operators on Hilbert space H, and $W(T)$ be a convex set. Then $W(T)$ is a closed set if and only if $W_{ess}(T) \subset W(T)$. In particular, if T is an n-tuple of compact operators and $W(T)$ is a convex set. Then $W(T)$ is a closed set if and only if $0 \in W(T)$. If $0 \bar{\in} W(T)$, then $0 \in \mathrm{Ext}\overline{W(T)}$, $\overline{W(T)} - W(T)$ contains only 0 or ray segments from 0 and these ray segments don't contain extreme points of $\overline{W(T)}$ but 0.

Proof The first assertion is clearly. To prove the second, we need only to prove $W_{ess}(T) = \{0\}$. Since $W_{ess}(T) \subseteq \bigcap_K \overline{W(T+K)} = \{0\}$, $W_{ess}(T)$ is a nonempmpty set. So $W_{ess}(T) = \{0\}$. Now we prove the third. If $0 \bar{\in} \mathrm{Ext}\overline{W(T)}$, by lemma2.14, we know there are $\lambda_1 \cdots \lambda_k \in \mathrm{Ext}\overline{W(T)}$, such that $0 \in \mathrm{con}\{\lambda_1 \cdots \lambda_k\}$, and $\lambda_i \neq 0 (i = 1, \cdots k)$, by theorem2.6, $\lambda_i \in W(T)$. Hence $0 \in W(T)$. It contradicts $0 \bar{\in} W(T)$.

Now suppose $0 \neq \lambda \in \overline{W(T)} - W(T)$, if $\lambda \in \mathrm{Ext}\overline{W(T)}$, then $\lambda \in W(T)$. So $\lambda \bar{\in} \mathrm{Ext}\overline{W(T)}$, hence there are $\lambda_1 \cdots \lambda_k$ in Ext $\overline{W(T)}$, such that $\lambda \in \mathrm{con}\{\lambda_1 \cdots \lambda_k\}$. If $\lambda_j \neq 0 (j = 1, 2, \cdots k)$, then $\lambda \in W(T)$ by theorem2.6, hence there is a number j such that $\lambda_j = 0$. Without loss of generality, we assume $\lambda_1 = 0$, and $\lambda_1 \neq 0$ $j = 2, \cdots k$. We'll prove $[0,\lambda] \cap W(T) = \phi$. In fact, if there is a λ_0 in $[0,\lambda] \cap W(T)$, let c be the extended line of $[0,\lambda]$, then $c \cap \mathrm{con}\{\lambda_2 \cdots \lambda_k\} \neq \phi$. Suppose $\lambda_1' \in c \cap \mathrm{con}\{\lambda_2 \cdots \lambda_k\}$, by $\lambda_j \neq 0$ $j = 2, \cdots, k$, we know $\lambda_1' \in \mathrm{con}\{\lambda_2 \cdots \lambda_k\} \subset W(T)$. Hence $\lambda \in \mathrm{con}\{\lambda_0, \lambda_1'\} \subset W(T)$. It contradicts the hypothesis.

Theorem 2.16 Let $T = (T_1 \cdots T_n)$ be an n-tuple of commuting operators on Hilbert space. $W(T)$ is a convex set, $\lambda \in \partial W(T)$, $\{f_k\}_1^\infty$ is a sequence of unit vectors of H such that $(Tf_k, f_k) = ((T_1 f_k, f_k) \cdots (T_n f_k, f_k)) \to \lambda$. If $\lambda \in \mathrm{Ext}\overline{W(T)}$, then either there is a sub-sequence $\{f_{kj}\}_{j=1}^\infty$ converges weakly to 0 or $\{f_{kj}\}_{j=1}^\infty$ converges weakly to $f \neq 0$ such that $(Tf, f) = \lambda \|f\|^2$.

Proof It is similar to theorem 2.6's.

Example 2.17 There is an n-tuple of commuting non-normal operators, $T = (T_1 \cdots T_n)$, such that the bare points of $W(T)$ are not the joint eigenvalues of T.

Let H be a 3-dimmensional Hilbert space, and $\{e_1, e_2, e_3\}$ be its orthogonal base

$$T_1 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \qquad T_2 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix},$$

then $T_1 T_2 = T_2 T_1 = 0$,

$$W(T_1 T_2) = \{(|a_2|^2, a_1 a_3) \mid |a_1|^2 + |a_2|^2 + |a_3|^2 = 1\}.$$

It is a circular cone in 3-dimmensional Euclidean space. We write the base of the circular cone by c. Clearly, all the points in ∂c are the bare points of $W(T_1 T_2)$, if $\partial c \subset \mathrm{Sp}_p(T, H)$. Let $\{x_\lambda\}_{\lambda \in \partial c}$ be the eigenvectors of T. Then $\{x_\lambda\}_{\lambda \in \partial c}$ is a linear indepence family. It is impossible because H is a 3-dimmensional space.

Remark Clearly, if T is a bounded linear operator on Hilbert space H, then the bare points of $W(T)$ may not belong to $\sigma_p(T)$. Hence, V.I.Istrǎtsescu [4]'s result is false (The result says if λ is a bare point of a bounded linear operator T, then $\lambda \in \sigma_p(T)$).

Acknowledgement *The author is indebted to Professor Zou Chengzu, Sun Yifeng and Sun Shanli for their help in the preparation of this article.*

References

[1] Cao Guangfu, Taylor's joint spectra and joint numerical ranges, Jilin University Master

Degree Thesis 1988.
[2] P.A.Fillmore, J.G.Stampfli and J.P.Williams, Acta, Sci.Math.Szeged 33,1972(179-192).
[3] R.E.Curto, Tran.Amer.Math. Vol. 266 No. 1 1981 129-159.
[4] V.I.Istrǎstescu, Introduction to linear operator theory, A Series of Monographs and Textbooks 65.

联合本质谱，联合本质数值域与联合数值域的边界

曹广福

（哈尔滨建筑工程学院）

摘要 本文研究了Hilbert空间上几乎交换有界线性算子组的联合本质谱，联合本质谱数值域与联合数值域的边界及其相互间的关系．主要结果有

定理1 设$T=(T_1\cdots T_n)$是Hilbert空间H上的几乎交换组，且$T_iT_j^*=T_j^*T_i\ (i\neq j)$，则

$$Sp_e(T,H)\subset W_{ess}(T).$$

定理2 设$T=(T_1\cdots T_n)$是Hilbert空间H上几乎交换本质正规组，则$Sp_e(T,H)\subset W_{ess}(T)$．

定理3 设$T=(T_1\cdots T_n)$是Hilbert空间H上几乎交换组，则$Sp_e(T,H)\subset \mathrm{con}W_{ess}(T)$．

定理4 设$T=(T_1\cdots T_n)$是Hilbert空间H上交换算子组，$W(T)$是凸集，则

$$\mathrm{Ext}\,\overline{W(T)}\subset W_{ess}(T)\cup W(T).$$

特别地，$W(T)=\mathrm{con}(W_{ess}(T)\cup W(T))$．

定理5 设$T=(T_1\cdots T_n)$是Hilbert空间H上交换算子组，$W(T)$是凸集，则$W(T)$闭当且仅当$W_{ess}(T)\subset W(T)$．特别地，如果T是紧算子组，$W(T)$是凸集，则$W(T)$是闭的当且仅当$0\in W(T)$．如果$0\in W(T)$，则$0\in \mathrm{Ext}\,\overline{W(T)}$，$\overline{W(T)}-W(T)$仅含0或从0出发的射线段，且这些射线段不含除0外的其它端点．