Journal of Mathematical Research and Exposition
Vol.10 No.1 Feb. 1990

The Extension of a Compact Set-Valued Measure*

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Abstract

In [1], the convexity, existence of selectors and Radon-Nikodym derivative of a set-valued measure have been developed. This paper is to invetigate the structure and extension of a compact set-valued measure.

§ 1 Set-Valued Measure and its Properties

Let R^m be the m-dimensional Euclidean space, $\mathscr{P}_0(R^m)$ ($\mathscr{K}_0$, co $\mathscr{K}_0$, respectively) the family of all nonempty subsets (nonempty compact subsets, nonempty compact convex subsets, respectively) of R^m. Let H denote the Hausdorff distance on $\mathscr{K}_0$, i.e. for K_1 and K_2 in $\mathscr{K}_0$.

$$H(K_1,K_2)=\max\{\max_{x\in K_1}\min_{y\in K_2} d(x,y), \max_{x\in K_2}\min_{y\in K_1} d(x,y)\}. \quad (1)$$

It is known that $(\mathscr{K}_0,H)$ is a complete metric space.

For $D\in \mathscr{P}_0(R^m)$, the norm of D is defined by

$$||D||=\sup\{\|x\|:x\in D\}. \quad (2)$$

For $D_n\in \mathscr{P}_0(R^m)$ $(n\geqslant 1)$, define the addition as follows[1]:

$$\sum_{n=1}^{\infty} D_n=\{\sum_{n=1}^{\infty}x_n:x_n\in D_n\ (n\geqslant 1),\ \sum_{n=1}^{\infty}\|x_n\|<\infty\}. \quad (3)$$

It is easy to verify that

$$H(K_1+K_2,K_2)\leqslant\|K_1\| \quad (4)$$

for K_1 and K_2 in $\mathscr{K}_0$.

Definition 1 Suppose $\mathscr{F}$ is an algebra on Ω. Mapping $\pi:\mathscr{F}\to\mathscr{P}_0(R^m)$ is called a set-valued measure on $\mathscr{F}$ if

$$\pi(\bigcup_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\pi(A_n) \quad (5)$$

for every sequence $\{A_n\}$ of mutually disjoint elements in $\mathscr{F}$ and $\bigcup_{n=1}^{\infty}A_n\in\mathscr{F}$. If $\|\pi(\Omega)\|<\infty$, then π is called bounded.

If π takes values in K_0 (coK_0), then π is called a compact (compact convex)

* Received June 25, 1988. The Project Supported by National Natural Science Foundation of China.

set-valued measure.

If π is a compact set-valued measure on $\mathscr{F}$, then the support function of π,

$$v_p(A)=S(p,\pi(A))=\sup\{px:x\in\pi(A)\}\ (p\in\mathbf{R}^m,\ A\in\mathscr{F}) \tag{6}$$

is a finite signed measure on $\mathscr{F}$. Let e_i $(i=\pm 1,\cdots,\pm m)$ be the $2m$ vectors $(0,\cdots,0,\pm 1,0,\cdots,0)$, $|v_{ei}|$ denote the total variation of v_{ei}. Denote

$$v(A)=\sum_{i=\pm 1}^{\pm m}|v_{ei}|(A).\quad (A\in\mathscr{F}).$$

v is called the tight-control measure of π and it is clear that for $A\in\mathscr{F}$

$$\|\pi(A)\|\leqslant v(A)\leqslant v(\Omega)<\infty. \tag{7}$$

Theorem 1 Let $\pi:\mathscr{F}\to K_0$, then π is a compact set-valued measure iff the following two conditions are satisfied:

(1) π is finitely additive,

(2) $H(\pi(A_n),\ \pi(A))\underset{n\to\infty}{\to}0$, whenever $\{A_n,A:n\geqslant 1\}\subset\mathscr{F}$ and $A_n\uparrow A$.

Proof "$\Rightarrow$". If π is a compact set-valued measure, then $\pi(\phi)=\{0\}$ π is finitely abbitive and $\pi(A)=\pi(A\setminus A_n)+\pi(A_n)$ whenever $A_n\uparrow A$. Hence

$$H(\pi(A),\ \pi(A_n))<\|\pi(A\setminus A_n\|\leqslant v(A\setminus A_n)\to 0\ (n\to\infty).$$

"$\Leftarrow$". Let $\{A_n:n\geqslant 1\}$ be a sequence of disjoint elements in $\mathscr{F}$ and $A=\bigcup_{n=1}^{\infty}A_n\in\mathscr{F}$, then

$$\lim_{n\to\infty}H(\pi(A),\ \sum_{i=1}^{n}\pi(A_i))=\lim_{n\to\infty}H(\pi(A),\ \pi(\bigcup_{i=1}^{n}A_i))=0.$$

Since π is finitely additive, it is easy to show that v is finitely additive. So for every $n\geqslant 1$,

$$\sum_{i=1}^{n}\|\pi(A_i)\|\leqslant\sum_{i=1}^{n}v(A_i)=v(\bigcup_{i=1}^{n}A_i)\leqslant v(\Omega)<\infty.$$

Thus $\sum_{n=1}^{\infty}\pi(A_n)$ is a nonempty bounded set. Noting that $\pi(A_n)\in K_0$, we get[7] $\sum_{n=1}^{\infty}\pi(A_n)\in K_0$. Therefore

$$H(\pi(A),\ \sum_{n=1}^{\infty}\pi(A_n))\leqslant H(\pi(A),\ \sum_{i=1}^{n}\pi(A_i))+\sum_{i=n+1}^{\infty}\|\pi(A_i)\|\to 0$$

i.e. $\pi(A)=\sum_{n=1}^{\infty}\pi(A_n)$.

§2 Extension of Compact Convex Set-Valued Measures

Let $\mathscr{F}$ be a algebra on Ω, $\sigma(\mathscr{F})$ the σ-algebra genrated by $\mathscr{F}$ (i.e. minimume σ-algebra containing $\mathscr{F}$) and π be a set-valued measure on $\mathscr{F}$. If there is a set-valued measure π' on $\sigma(\mathscr{F})$ such that $\pi'(A)=\pi(A)$ $(A\in\mathscr{F})$ then π' is called an extension of π from $\mathscr{F}$ to $\sigma(\mathscr{F})$.

Definition 2 Let $\mu:\mathscr{F}\to\mathrm{R}^m$, $\mu_t:\mathscr{F}\to\mathrm{R}^m$ $(t\in T)$ are singned vector measures on $(\Omega,\mathscr{F})$, μ is called strong additive if

$$\|\sum_{n=1}^{\infty}\mu(A_n)\|<\infty \tag{8}$$

for every sequence $\{A_n\}$ of mutuall y disjoint elements in $\mathscr{F}$. $\{\mu_t:t\in T\}$ is called uniformly strong additive if $\{\mu_t:t\in T\}$ are strong additive and

$$\lim_{n\to\infty}\sup_{t\in T}\|\sum_{l=n}^{\infty}\mu_t(A_l)\|=0$$

for sequence $\{A_n\}$ of mutually disjoint elments in $\mathscr{F}$. $\{\mu_t:t\in T\}$ is called uniformly bounded if for $A\in\mathscr{F}$.

$$\sup_{t\in T}\|\mu_t(A)\|<\infty \tag{10}$$

It is easy to verify that statement (9) is equivalent to for $A_n\in\mathscr{F}$ $(n\geqslant 1)$, $A_n\uparrow\varnothing$:

$$\lim_{n\to\infty}\sup_{t\in T}\|\mu_t(A_n)\|=0. \tag{11}$$

Let $(\Omega,\mathscr{A})$ be a measurable space, π a bounded set-valued measure on $(\Omega,\mathscr{A})$. Z.Artstein[1] proved that for arbitrary $A\in\mathscr{A}$ and $x\in\pi(A)$ there exists a vector measure μ on $(\Omega,\mathscr{A})$ such that $\mu(A)\in\pi(A)$ $(A\in\mathscr{A})$ and $\mu(A)=x$; this μ is called a measure selector of π. With little modification the proof is also suitable for the case that algebra $\mathscr{F}$ replaces σ-algebra $\mathscr{A}$, i.e. if $\mathscr{F}$ is an algebra and π is is a bounded set-valued measure on $\mathscr{F}$, then for arbitrary $A\in\mathscr{F}$ and $x\in\pi(A)$, there exists a vector measure μ on $(\Omega,\mathscr{F})$ such that $\mu(A)\in\pi(A)$ $(A\in\mathscr{F})$ and $\mu(A)=x$.

Theorem 2 Let $\mathscr{F}$ be an algebra on Ω, then π is a compact convex set-valued measure on $\mathscr{F}$ iff there exists a sequence $\{\mu_n\}$ of uniformly bounded and uniformly strong additive signed measures on $\mathscr{F}$ such that

$$\pi(A)=\overline{\mathrm{co}}\{\mu_n(A):n\geqslant 1\}\ (A\in\mathscr{F}) \tag{12}$$

and π is finitely additive.

Proof "$\Leftarrow$". By (12)

$$\|\pi(A)\|=\|\overline{\mathrm{co}}\{\mu_n(A):n\geqslant 1\}\|\leqslant\sup_{n\geqslant 1}\|\mu_n(A)\|<\infty$$

i.e. $\pi(A)\in\mathrm{co}K_0$. Since $\{\mu_n\}$ are uniformly strong additive, fo: arbitrary A, $A_k\in\mathscr{F}$ $(k\geqslant 1)$, $A_k\uparrow A$ from

$$H(\pi(A_k),\pi(A))\leqslant\|\pi(A\setminus A_k)\|\leqslant\sup\|\mu_n(A\setminus A_k)\|\to 0\ (k\to\infty)$$

and Theorem 1 it follows that π is a compact convex set-valued measure on $\mathscr{F}$.

"$\Rightarrow$" If π is a compact convex set-valued measure on $\mathscr{F}$, then π is finitely additive. Since $\pi(\Omega)\in\mathrm{co}K_0$, there is a countable set $D=\{x_1,x_2,\cdots\}$ of $\pi(\Omega)$ which is dense in $\pi(\Omega)$. For $x_n\in D$ there exists selector μ_n of π on $\mathscr{F}$ such that $\mu_n(\Omega)=x_n\,(n\geqslant 1)$. Hence

$$\pi(\Omega)=\mathrm{cl}\,D=\mathrm{cl}\{\mu_n(\Omega):n\geqslant 1\}. \tag{13}$$

By the convexity of $\pi(\Omega)$ it yields

$$\pi(\Omega)=\overline{\mathrm{co}}\{\mu_n(\Omega):n\geqslant 1\}. \tag{14}$$

For arbitrary $A\in\mathscr{F}$,

$$\begin{aligned}\pi(\Omega)&=\overline{\mathrm{co}}\{\mu_n(\Omega):n\geqslant 1\}\\&\subset\overline{\mathrm{co}}\{\mu_n(A):n\geqslant 1\}+\overline{\mathrm{co}}\{\mu_n(A^c):n\geqslant 1\}\\&\subset\pi(A)+\pi(A^c)=\pi(\Omega)\end{aligned}$$

so that

$$\pi(A)+\pi(A^c)=\overline{\mathrm{co}}\{\mu_n(A):n\geqslant 1\}+\overline{\mathrm{co}}\{\mu_n(A^c):n\geqslant 1\}.$$

Noting that $\overline{\mathrm{co}}\{\mu_n(A):n\geqslant 1\}\subset\pi(A)$, $\overline{\mathrm{co}}(\mu_n(A^c):n\geqslant 1\}\subset\pi(A^c)$, we get that

$$\pi(A)=\overline{\mathrm{co}}\{\mu_n(A):n\geqslant 1\}\quad(A\in\mathscr{F}). \tag{15}$$

It is obvious that $\|\mu_n(A)\|\leqslant\|\pi(A)\|<\infty$ $(n\geqslant 1)$ for every $A\in\mathscr{F}$, hence

$$\sup_{n\geqslant 1}\|\mu_n(A)\|\leqslant\|\pi(A)\|<\infty$$

i.e. $\{\mu_n:n\geqslant 1\}$ are uniformly bounded. For sequence $\{A_i\}$ of mutually disjoint elements in $\mathscr{F}$.

$$\|\sum_{i=1}^{\infty}\mu_n(A_i)\|\sum_{i=1}^{\infty}v(A_i)\leqslant v(\Omega)<\infty$$

i.e. $\{\mu_n\}$ are strong additive. If $A_k\in\mathscr{F}$ $(k\geqslant 1)$, $A_k\downarrow\varnothing$, then

$$\sup_{n\geqslant 1}\|\mu_n(A_k)\|\leqslant\|\pi(A_k)\|\leqslant v(A_k)\to 0$$

i.e. $\{\mu_n\}$ are uniformly strong additive.

Theorem 3 Let π be a compact convex set-valued measure on algebra $\mathscr{F}$ and $\sigma(\mathscr{F})$ the σ-algebra generated by $\mathscr{F}$, then there uniquely exists a compact convex set-valued measure $\bar{\pi}$ on $\sigma(\mathscr{F})$ such that $\bar{\pi}(A)=\pi(A)$ $(A\in\mathscr{F})$. This $\bar{\pi}$ is called the extension set-valued measure of π.

Proof If π is a compact convex set-valued measure on $(\Omega,\mathscr{F})$, by Theorem 2 there exists a sequence $\{\mu_n\}$ of uniformly bounded and uniformly strong additive signed measures on $\mathscr{F}$ such that

$$\pi(A)=\overline{\mathrm{co}}\{\mu_n(A):n\geqslant 1\}\quad(A\in\mathscr{F}). \tag{16}$$

Let $\bar{\mu}_n$ be the extension of μ_n to $\sigma(\mathscr{F})$ $(n\geqslant 1)$, and denote

$$\bar{\pi}(A)=\overline{\mathrm{co}}\{\bar{\mu}_n(A):n\geqslant 1\}\quad(A\in\sigma(\mathscr{F})). \tag{17}$$

It is clear that $\bar{\pi}(A)=\pi(A)$ $(A\in\mathscr{F})$.

Let v be the tight-control measure of π on $\mathscr{F}$ and $\bar{v}$ the extension of v to $\sigma(\mathscr{F})$, then $\bar{v}$ is finite since v is finite. Put

$$\mathscr{C}=\{A:\|\bar{\mu}_n(A)\|\leqslant\bar{v}(A)\ (n\geqslant 1)\}.$$

It is obvious that $\mathscr{F}\subset\mathscr{C}$. For arbitrary $A_k\in\mathscr{C}(k\geqslant 1)$ $A_k\uparrow(\downarrow A)$,

$$\|\bar{\mu}_n(A)\|=\|\lim_{k\to\infty}\bar{\mu}_n(A_k)\|\leqslant\lim_{k\to\infty}\bar{v}(A_k)=\bar{v}(A).$$

Hence $A\in\mathscr{C}$, i.e. $\mathscr{C}$ is a monotone class. Thus $\sigma(\mathscr{F})\subset\mathscr{C}$ by the monotone class theorem, i.e.

$$\|\overline{\mu}_n(A)\| \leqslant \overline{v}(A) \quad (n \geqslant 1, \ A \in \sigma(\mathscr{F})). \tag{18}$$

By (17)

$$\|\overline{\pi}(A)\| \leqslant \overline{v}(A) \tag{19}$$

i.e. $\overline{\pi}(A) \in \mathrm{co}K_0$ $(A \in \sigma(\mathscr{F}))$ and $\{\overline{\mu}_n : n \geqslant 1\}$ are uniformly bounded on $\sigma(\mathscr{F})$. By (19) in a way similar to Theorem 2 we can show that $\{\overline{\mu}_n : n \geqslant 1\}$ are uniformly strong additive on $\sigma(\mathscr{F})$. By Theorem 2, to prove $\overline{\pi}$ is a compact convex set-valued measure on $\sigma(\mathscr{F})$ it suffices to prove $\overline{\pi}$ is finitely additive.

Firstly, since $\{\overline{\mu}_n : n \geqslant 1\}$ are uniformly strong additive, for $A_k \uparrow (\downarrow) A$ we have

$$\begin{aligned} H(\overline{\pi}(A), \overline{\pi}(A_k)) &= H(\mathrm{co}\{\overline{\mu}_n(A) : n \geqslant 1\}, \mathrm{co}\{\overline{\mu}_n(A_k) : n \geqslant 1\}) \\ &= H(\mathrm{co}\{\overline{\mu}_n(A_k) + \overline{\mu}_n(A \setminus A_k) : n \geqslant 1\}, \mathrm{co}\{\overline{\mu}_n(A_k) : n \geqslant 1\}) \\ &\leqslant \sup_{n \geqslant 1} \|\overline{\mu}_n(A \setminus A_k)\| \to 0 \ (k \to \infty). \end{aligned}$$

i.e. for $A_k \uparrow (\downarrow) A$,

$$\overline{\pi}(A) = \lim_{k \to \infty} \overline{\pi}(A_k) \tag{20}$$

For $A \in \mathscr{F}$, denote

$$\mathscr{C}_1 = \{B : \overline{\pi}(A \cup B) = \overline{\pi}(A) + \overline{\pi}(B) : B \cap A = \varnothing\}.$$

It is clear that $\mathscr{F} \cap A^c \subset \mathscr{C}_1$. For $B_k \in \mathscr{C}_1$ $(k \geqslant 1)$ and $B_k \uparrow B$ it is obvious that $B \cap A = \varnothing$ and

$$\begin{aligned} \overline{\pi}(B \cup A) &= \overline{\pi}\Big(\bigcup_{k=1}^{\infty} (B_k \cup A)\Big) \\ &= \lim_{k \to \infty} \overline{\pi}(B_k \cup A) = \lim_{k \to \infty} (\overline{\pi}(B_k) + \pi(A)) \\ &= \lim_{k \to \infty} \overline{\pi}(B_k) + \pi(A) = \overline{\pi}(B) + \pi(A). \end{aligned}$$

Hence $B \in \mathscr{C}_1$. Similarly we can prove that $B \in \mathscr{C}_1$ whenever $B_k \in \mathscr{C}_1$ $(k \geqslant 1)$ and $B_k \downarrow B$. So $\mathscr{C}_1$ is a monotone class. Thus $\sigma(\mathscr{F}) \cap A^c \subset \mathscr{C}_1$, i.e. for arbitrary $A \in \mathscr{F}$ and $B \in \sigma(\mathscr{F})$ if $A \cap B = \varnothing$ then $\overline{\pi}(A \cup B) = \overline{\pi}(A) + \overline{\pi}(B)$.

For $B \in \sigma(\mathscr{F})$, denote $\mathscr{C}_2 = \{A : \overline{\pi}(A \cup B) = \overline{\pi}(A) + \overline{\pi}(B), \ A \cap B = \varnothing\}$.

Similarly to the above process we can prove that $\mathscr{C}_2$ is a monotone class and $\mathscr{F} \cap B^c \subset \mathscr{C}_2$. Hence $\sigma(\mathscr{F}) \cap B^c \subset \mathscr{C}_2$. i.e. for A and B in $\sigma(\mathscr{F})$ if $A \cap B = \varnothing$ then $\overline{\pi}(A \cup B) = \overline{\pi}(A) + \overline{\pi}(B)$. By Theorem 2, $\overline{\pi}$ is a compact convex set-valued measure.

Suppose π^* is another compact convex set-valued measure on $\sigma(\mathscr{F})$ such that $\pi^*(A) = \pi(A)$ $(A \in \mathscr{F})$. Denote $\mathscr{C}_3 = \{A : \overline{\pi}(A) = \pi^*(A)\}$, then $\mathscr{F} \subset \mathscr{C}_3$. By (20) it is straightforward that $\mathscr{C}_3$ is a monotone class, so $\sigma(\mathscr{F}) \subset \mathscr{C}_3$ i.e. $\overline{\pi}(A) = \pi^*(A)$ for each $A \in \sigma(\mathscr{F})$. Thus the extension measure is unique. Now the proof is completed.

§ 3 Extension of Compact Set-valued Measures

To study the structure and extension of compact set-valued measure, we introduce the concept of atoms of set-valued measures.

Definition 3 Let $\mathscr{F}$ be an algebra on Ω and π a set-valued measure on $\mathscr{F}$, $A\in\mathscr{F}$ is called an atom of π if $\pi(A)\neq\{0\}$ and $\pi(B)=\{0\}$ or $\pi(A\setminus B)=\{0\}$ whenever $B\subset A$ and $B\in\mathscr{F}$. If π has no atoms, then π is called nonatomic. Given measure μ on $\mathscr{F}$, π is called absolutely continuous with respect to μ if $\pi(A)=\{0\}$ whenever $A\in\mathscr{F}$, $\mu(A)=0$; and denoted it by $\pi\ll\mu$.

Let v be the tight-control measure of π, it is obvious that $\pi\ll\mu$.

Theorem 4 A is an atom of π iff A is an atom of v.

Proof If $A\in\mathscr{F}$ is an atom of π, then $\pi(A)\neq\{0\}$ and $\pi(B)=\{0\}$ or $\pi(A\setminus B)=\{0\}$ whenever $B\in\mathscr{F}$, $B\subset A$. Hence $v_{ei}(B)=0$ $(i=\pm1,\cdots,\pm m)$ or $v_{ei}(B)=v_{ei}(A)$ $(i=\pm1,\cdots,m)$. By Hahn decomposition

$$v_{ei}(B)=v_{ei}^{+}(B)-v_{ei}^{-}(B),\quad |v_{ei}|(B)=v_{ei}^{+}(B)+v_{ei}^{-}(B).$$

Where

$$v_{ei}^{+}(B)=\sup\{v_{ei}(D):D\subset B,\ D\in\mathscr{F}\},\quad v_{ei}^{-}(B)=-\inf\{v_{ei}(D):D\subset B,\ D\in\mathscr{F}\}.$$

If $v_{ei}(A)>0$, then $v_{ei}^{+}(B)=0$ or $v_{ei}^{+}(B)=v_{ei}(A)$ $(B\subset A)$, $v_{ei}^{-}(B)=0$. Hence we get $|v_{ei}|(B)=|v_{ei}(B)|$ $(B\subset A)$. This result holds also for $v_{ei}(A)<0$. Thus $v(B)=\sum_{i=\pm1}^{\pm m}|v_{ei}(B)|$. From $\pi(B)=\{0\}$ it follows that $v_{ei}(B)=0$ $(i=\pm1,\cdots,\pm m)$, so that $v(B)==0$. Similarly $v(A\setminus B)=0$ whenever $\pi(A\setminus B)=\{0\}$. By $\pi\ll v$ and $\pi(A)\neq\{0\}$, $v(A)\neq0$. Therefore A is an atom of v.

If $A\in\mathscr{F}$ and A is an atom of v, then $v(A)\neq0$ and $v(B)=0$ or $v(A\setminus B)=0$ whenever $B\subset A$. By $\pi\ll v$, it yields $\pi(A)=\{0\}$ or $\pi(A\setminus B)=\{0\}$. Similarly to the above process we can show that

$$v(B)=\sum_{i=\pm1}^{\pm m}|v_{ei}(B)|\quad(B\subset A).$$

Hence by $v(A)\neq0$, $\pi(A)\neq\{0\}$, i.e. A is an atom of π.

Theorem 5 If $\mathscr{F}$ is an algebra on Ω and π is a nonatomic compact set-valued measure on $\mathscr{F}$, then π is compact convex on $\mathscr{F}$.

Proof It suffices to show that $\pi(\Omega)$ is a convex set. Let x and y be in $\pi(\Omega)$, then there exist vector measures μ_1 and μ_2 on $\mathscr{F}$ such that $\mu_1(A)\in\pi(A)$, $\mu_2(A)\in\pi(A)$ $(A\in\mathscr{F})$ and $\mu_1(\Omega)=x$, $\mu_2(\Omega)=y$. Put $\mu=\binom{\mu_1}{\mu_2}$ then μ is a nonatomic bounded vector measure on $\mathscr{F}$. Let $\overline{\mu}$ be the extension of μ to $\sigma(\mathscr{F})$, then $\overline{\mu}$ is also nonatomic bounded vector measure. By [3], $\overline{\mu}$ is convev, so that for $0<\lambda<1$ there is $A\in\sigma(\mathscr{F})$ such that $\overline{\mu}(A)=\lambda\mu(\Omega)$, i.e. $\overline{\mu_1}(A)=\lambda x$ and $\overline{\mu_2}(A)=\lambda y$. By the extension theorem of measures there exist $A_k\in\mathscr{F}$ $(k\geq1)$, $A_k\downarrow$, $\overline{\mu}(A)=\lim_{k\to\infty}\overline{\mu}(A_k)=\lim_{k\to\infty}\mu(A_k)$, hence

$$\lambda x+(1-\lambda)y=\overline{\mu_1}(A)+\overline{\mu_2}(\Omega\setminus A)$$

$$=\lim_{k\to\infty}(\mu_1(A_k)+\mu_2(\Omega\setminus A_k))$$

$$\in\pi(A_k)+\pi(\Omega\setminus A_k)=\pi(\Omega)$$

i.e. $\pi(\Omega)$ is a convex set.

Theorem 6 Let $\mathscr{F}$ be an algebra on Ω, π a compact set-valued measure on $\mathscr{F}$, then there exist at most countable mutually disjoint atoms $\{B_i: i\geqslant 1\}$ of π such that

$$\pi_1(A)=\lim_{n\to\infty}\pi(A\cap(\bigcup_{i=1}^{n}B_i))\quad(A\in\mathscr{F})\tag{21}$$

and

$$\pi_2(A)=\lim_{n\to\infty}\pi(A\cap(\bigcup_{i=1}^{n}B_i)^c)\quad(A\in\mathscr{F})\tag{22}$$

are compact set-valued measure and compact convex set-valued measure respectively, and

$$\pi(A)=\pi_1(A)+\pi_2(A)\quad(A\in\mathscr{F}).\tag{23}$$

Proof Since the tight-control measure v of π is finite, the mutually disjoint atoms are at most countable, denoted them by $B_i\,(i\geqslant 1)$. By Theorem 4, $\{B_i\}$ are atoms of π. It is easy to show that π_1 and π_2 are finitely additive, and for $A_k\in\mathscr{F}$, $A\in\mathscr{F}$ when $A_k\uparrow A$,

$$H(\pi_1(A_k),\pi_1(A))=H(\lim_{n\to\infty}\pi(A_k\cap(\bigcup_{i=1}^{n}B_i)),\lim_{n\to\infty}\pi(A\cap(\bigcup_{i=1}^{n}B_i)))$$

$$=\lim_{n\to\infty}H(\pi(A_k\cap(\bigcup_{i=1}^{n}B_i)),\pi(A\cap(\bigcup_{i=1}^{n}B_i)))$$

$$<\|\pi(A\setminus A_k)\|<v(A\setminus A_k)\to 0\quad(k\to\infty)$$

i.e. $\lim_{k\to\infty}\pi_1(A_k)=\pi_1(A)$. By Theorem 1, π_1 is a compact set-valued measure on $\mathscr{F}$. From

$$\sum_{n=1}^{\infty}\|\pi(A\cap B_n)\|<\sum_{n=1}^{\infty}v(A\cap B_n)\leqslant v(A)<\infty$$

it follows $\sum_{n=1}^{\infty}\pi(A\cap B_n)\in K_0$ and

$$\pi_1(A)=\sum_{n=1}^{\infty}\pi(A\cap B_n)\tag{24}$$

Similarly it may be proved that π_2 is a compact set-valued measure and is nonatomic, so that π_2 is compact convex by Theorem 5. Obviously (23) holds.

Theorem 7 Let $\{A_n\}$ be a mutually disjoint sequence in $\mathscr{F}$, $\Omega=\bigcup_{n=1}^{\infty}A_n$ and $\mathscr{F}$ be the minimue algebra containing $\{A_n: n\geqslant 1\}$, π is a compact set-valued measure on $\mathscr{F}$ iff there exist a sequence $\{\mu_l: l\geqslant 1\}$ of uniformly bounded and uniformly strong additive measure selectors of π such that

$$\pi(B)=\mathrm{cl}\{\mu_l(B): l\geqslant 1\}\quad (B\in\mathscr{F}) \tag{25}$$

and π is finitely additive.

Proof Let D_n be a countable dense subset of $\pi(A_n)$ and $D=\bigcup_{n=1}^{\infty}D_n$, then D is also countable. For each $x\in D_n$ there exists selector μ of π on $\mathscr{F}$ such that $\mu(A_n)=x$. Hence there are countable measures $\{\mu_l\}$ such that (25) holds. By $\|\pi\|\ll v$, it is easy to prove that $\{\mu_l: l\geqslant 1\}$ are uniformly bounded and uniformly strong additive. Also by $\pi(\varnothing)=\{0\}$ it is obvious that π is finitely additive. The sufficiency is similar to that of Theorem 2.

Theorem 8 Let $\mathscr{F}$ be an algebra on Ω and π a compact set-valued measure on $(\Omega, \mathscr{F})$, then there uniquely exists a compact set-valued measure $\bar{\pi}$ on $\sigma(\mathscr{F})$ such that

$$\bar{\pi}(A)=\pi(A)\quad (A\in\mathscr{F})$$

Proof By Theorem 6 there exist at most countable atoms $\{B_i: i\geqslant 1\}$ of π such that

$$\pi(A)=\pi_1(A)+\pi_2(A)\quad (A\in\mathscr{F}).$$

Where $\pi_1(A)=\sum_{n=1}^{\infty}\pi(A\cap B_n)$ and π_2, defined in Theorem 6, are compact set-valued measure and compact convex set-valued measure respectively. By Theorem 3, there exists compact convex set-valued $\bar{\pi}_2$ on $\sigma(\mathscr{F})$ such that

$$\bar{\pi}_2(A)=\pi_2(A)\quad (A\in\mathscr{F}).$$

By Theorem 7 in a manner similar to Theorem 3 we can show that there is a compact set-valued measure $\bar{\pi}_1$ on $\sigma(\mathscr{F})$ such that $\bar{\pi}_1(A)=\pi_1(A)$ $(A\in\mathscr{F})$. Put

$$\bar{\pi}(A)=\bar{\pi}_1(A)+\bar{\pi}_2(A)\quad (A\in\sigma(\mathscr{F}))$$

then $\bar{\pi}$ is a compact set-valued measure on $\sigma(\mathscr{F})$ and $\bar{\pi}(A)=\pi(A)$ $(A\in\mathscr{F})$. Similarly to Theorem 3 the extension is proved to be unique.

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紧 集 值 测 度 的 扩 张*

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摘　　要

本文研究了紧集值测度的结构特征与扩张，给出如下主要结果：

（1）设$\mathscr{F}$是Ω上的集代数，则π是$\mathscr{F}$上的紧凸集值测度的充要条件是在$\mathscr{F}$上存在一列一致有界，一致强可加的广义测度$\{\mu_n : n\geqslant 1\}$使$\pi(A)=\overline{\mathrm{co}}\{\mu_n(A):n\geqslant 1\}$（$A\in\mathscr{F}$）且$\pi$是有限可加的.

（2）设π是$\mathscr{F}$上的紧凸集值测度，$\sigma(\mathscr{F})$为$\mathscr{F}$生成的σ-代数，则在$\sigma(\mathscr{F})$上存在唯一的紧凸集值测度$\overline{\pi}$使$\overline{\pi}(A)=\pi(A)$（$A\in\mathscr{F}$）.

该结果证明思路：利用（1）将π分解为$\pi(A)=\overline{\mathrm{co}}\{\mu_n(A):n\geqslant 1\}$（$A\in\mathscr{F}$）；将$\mu_n$扩张到$\sigma(\mathscr{F})$上，记为$\overline{\mu}_n(n\geqslant 1)$，定义$\overline{\pi}(A)=\overline{\mathrm{co}}\{\overline{\mu}_n(A):n\geqslant 1\}$（$A\in\sigma(\mathscr{F})$），先证明$\{\overline{\mu}_n\}$是一致有界，一致强可加，然后通过证明

$$\mathscr{C}_1=\{B:\overline{\pi}(A\cup B)=\overline{\pi}(A)+\overline{\pi}(B),\ B\cap A=\varnothing\}\quad (A\in\mathscr{F})$$

$$\mathscr{C}_2=\{A:\overline{\pi}(A\cup B)=\overline{\pi}(A)+\overline{\pi}(B),\ A\cap B=\varnothing\}\quad (B\in\sigma(\mathscr{F})).$$

是单调类，可得$\overline{\pi}$在$\sigma(\mathscr{F})$上是有限可加的．由（1），$\overline{\pi}$是π在$\sigma(\mathscr{F})$上的扩张.

（3）利用集值测度的原子集，将π分解为紧凸部分与可数集类上的紧部分，然后分别将之扩张，可得欲证的扩张.