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Some Remarks on the k-Uniformly Convexity and k-Uniformly Smoothness*

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Introduction. In [2], V.I.Istratescu discussed the k-uniformly convex space. Let us recall that a Banach space X is said to be k-uniformly convex ($k>1$) if for every $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that if $x_1,\cdots,x_k$ are vectors in the closed unit ball with $\|x_n-x_m\|\geq\varepsilon$ for $n\neq m$, then $\|x_1+\cdots+x_k\|\leq k(1-\delta(\varepsilon))$. In [4], Jong Sook Bae and Sung Kyu Choi have proved that the k-uniformly convexity is equivalent to uniformly convexity, hence the k-uniformly convexity is not a new notion. In [3], Istratescu also introduced another k-uniformmly convexity. A Banach space is called the k-uniformly cokvex if for every $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that whenever (x_i), (y_i), $i=1,\cdots,k$, are vectors in the closed unit ball of X and $\sum_{i=1}^{k}\|x_i-y_i\|\geq\varepsilon$, then

$$\|x_1+\cdots+x_k+y_1+\cdots+y_k\|\leq 2k(1-\delta(\varepsilon)).$$

The function $\delta_{k,X}(t)=\inf\{1-(1/2k)\|x_1+\cdots+x_k+y_1+\cdots+y_k\|:\ \sum_{i=1}^{k}\|x_i-y_i\|\geq t\}$ is called the k-modulus of convexity of X.

Let $2k\rho_{k,X}(s)=\sup\{\sum_{i=1}^{k}(\|x+sy_i\|+\|x-sy_i\|)-2k;\ \|x\|=\|y_i\|=1\}$, the function $\rho_{k,X}(s)$ is called the k-modulus of smoothness of X.

A Banach space X is called k-uniformly smooth if $\rho_{k,X}(s)\to 0$ as $s\to 0$. Istratescu have proved that for any Banach space the following relation holds

$$\rho_{k,X^*}(s)=\sup\{\frac{st}{2k}-\delta_{k,X}(t)\}$$

In this note, we prove that the above k-uniformly convexity also equivalent to uniformly convexity, and x is k-uniformly smooth if and only if X is uniformly smooth.

Theorem 1 Let X be a Banach space, then X is k-uniformly convex if and only if X is uniformly convex.

Proof If X is a uniformly convex Banach space, $\varepsilon>0, x_1,\cdots,x_k, y_1,\cdots,y_k$

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are in $U(X)=\{x\in X: \|x\|\leqslant 1\}$ and $\sum_{i=1}^{k}\|x_i-y_i\|\geqslant\varepsilon$, then there exists i_0 (we may assume that $i_0=1$) such that $\|x_1-y_1\|\geqslant\varepsilon/k$. For the ε/k, by the uniformly convexity of X, there is a $\delta'(\varepsilon/k)=\delta(\varepsilon)$ such that $\|x_1+y_1\|\leqslant 2(1-\delta(\varepsilon))$, therefore

$$\|x_1+\cdots+x_k+y_1+\cdots+y_k\|\leqslant\|x_1+y_1\|+\|x_2\|+\cdots+\|x_k\|+\|y_2\|+\cdots+\|y_k\|$$
$$\leqslant 2(1-\delta(\varepsilon))+2k-2=2k(1-\delta(\varepsilon)/2k).$$

Hence X is k-uniformly convex.

Conversely, if X is k-uniformly convex and $\varepsilon>0$ is given. Let $x, y\in U(X)$, $\|x-y\|\geqslant\varepsilon$. Take $x_1=x_2=\cdots=x_k=x$, $y_1=y_2=\cdots=y_k=y$, then $\sum_{i=1}^{k}\|x_i-y_i\|\geqslant k\varepsilon$, by the k-uniformly convexity of X, we have

$$k\|x+y\|=\|x_1+\cdots+x_k+y_1+\cdots+y_k\|\leqslant 2k(1-\delta(k\varepsilon))$$

so

$$\|x+y\|\leqslant 2(1-\delta(k\varepsilon)),$$

This prove that X is uniformly convex.

Theorem 2 Let X be a Banach space, $\delta_X(\varepsilon)$ and $\delta_{k,X}(\varepsilon)$ be respectively the modulus and k-modulus of convexity of X, then

$$\delta_X(\varepsilon)/k\leqslant\delta_{k,X}(k\varepsilon)\leqslant\delta_X(\varepsilon).$$

Proof Let $\varepsilon>0, x_1, y_1\in U(X)$, then for any $\{x_2,\cdots,x_k,y_2,\cdots,y_k\}\subset U(X)$, $\sum_{i=1}^{k}\|x_i-y_i\|\geqslant k\varepsilon$ implies

$$\delta_{k,X}(k\varepsilon)\leqslant 1-(1/2k)\|x_1+\cdots+x_k+y_1+\cdots+y_k\|$$

In particular, for $x_2=\cdots=x_k=x_1$, $y_2=\cdots=y_k=y_1$, we have

$$\delta_{k,X}(k\varepsilon)\leqslant 1-(1/2k)\|kx_1+ky_1\|=1-\frac{1}{2}\|x_1+y_1\|$$

Thus

$$\delta_{k,X}(k\varepsilon)\leqslant\inf\{1-\frac{1}{2}\|x+y\|: x, y\in U(X),\ \|x-y\|\geqslant\varepsilon\}=\delta_X(\varepsilon).$$

Now assume that $x_1,\cdots,x_k,y_1,\cdots,y_k$ are in $U(X)$ and $\sum_{i=1}^{k}\|x_i-y_i\|\geqslant k\varepsilon$. Then there exists i (we may assume that $i=1$) such that $\|x_1-y_1\|\geqslant\varepsilon$. Note that

$$1+\frac{1}{2}\|x_1+\cdots+x_k+y_1+\cdots+y_k\|\leqslant 1+\frac{1}{2}\|x_1+y_1\|+$$
$$\frac{\|x_2\|+\cdots+\|x_k\|+\|y_2\|+\cdots+\|y_k\|}{2}\leqslant 1+\frac{1}{2}\|x_1+y_1\|+\frac{2k-2}{2}=k+\frac{1}{2}\|x_1+y_1\|,$$

which implies

$$\frac{1}{k}(1-\frac{1}{2}\|x_1+y_1\|)\leqslant 1-\frac{1}{2k}\|x_1+\cdots+x_k+y_1+\cdots+y_k\|$$

therefore

$$\frac{1}{k}\delta_X(\varepsilon)\leqslant 1-\frac{1}{2k}\|x_1+\cdots+x_k+y_1+\cdots+y_k\|$$

Thus

$$\frac{1}{k}\delta_X(\varepsilon)\leqslant\delta_{k,X}(k\varepsilon),$$

the proof is end.

We recall that a Banach space X is said to be uniformly smooth if for every $\varepsilon>0$ there exists $\delta>0$ such that if $x,y\in X$ $\|x\|=1$ and $\|y\|<\delta$ then

$$\|x+y\|+\|x-y\|<2+\varepsilon\|y\|$$

The function $\rho_X(\tau)=\sup\{\|\frac{x+y}{2}\|+\|\frac{x-y}{2}\|-1:\|x\|=1,\|y\|=\tau\}$ is called the modulus of smoothness of X. It is well known that X is uniformly smooth if and only if $\rho_X(\tau)/\tau\to 0$ as $\tau\to 0$.

Theorem 3 For any Banach space X, we have

$$\rho_X(\tau)=\rho_{k,X}(\tau).$$

Proof Let $\tau>0, x, y\in X$, $\|x\|=1$ and $\|y\|=\tau$. Set $z_1=y/\tau$, then $\|z_1\|=1$ Now take $z_2=\cdots z_k=z_1$, then

$$\sum_{i=1}^{k}(\|x+\tau z_i\|+\|x-\tau z_i\|)-2k=k(\|x+y\|+\|x-y\|-2)$$

$$=2k(\|\frac{x+y}{2}\|+\|\frac{x-y}{2}\|-1)$$

hence

$$2k\rho_{k,X}(\tau)\geqslant 2k(\|\frac{x+y}{2}\|+\|\frac{x-y}{2}\|-1).$$

By arbitrariness of x and y, we obtain the inequality

$$2k\rho_{k,X}(\tau)\geqslant 2k\rho_X(\tau),$$

that is $\rho_{k,X}(\tau)\geqslant\rho_X(\tau)$. Now assume that $\|x\|=\|y_i\|=1, i=1,2,\cdots,k,\tau>0$. Set

$$\sum_{i=1}^{k}(\|x+\tau y_i\|+\|x-\tau y_i\|)-2k=a,$$

then there exists j, $1\leqslant j\leqslant k$, such that

$$\|x+\tau y_j\|+\|x-\tau y_j\|-2\geqslant a/k,$$

so

$$\|\frac{x+\tau y_j}{2}\|+\|\frac{x-\tau y_j}{2}\|-1\geqslant a/2k.$$

Let $\tau y_j=z$, then $\|z\|=1$ and $\|\frac{x+z}{2}\|+\|\frac{x-z}{2}\|-1\geqslant a/2k$, it follows that

$$\rho_X(\tau)\geqslant a/2k,$$

hence

$$\rho_X(\tau) \geqslant \rho_{k,X}(\tau).$$

Thus we obtain

$$\rho_X(\tau) = \rho_{k,X}(\tau).$$

Theorem 4 Let X be a Banach space, then X is k-uniformly smooth if and only if X is uniformly smooth.

Proof. This is an immediate consequence of Theorem 3.

References

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关于 k-致凸性和 k-致光滑性的几点注记

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摘要 设X为Banach空间，记$U(X)=\{x\in X: \|x\|\leqslant 1\}$. V.I.Istratescu引入了下面两个概念.Banach空间Z叫做k一致凸的，如果对每个$\varepsilon>0$，存在$\delta(\varepsilon)>0$，当$x_1,\cdots,x_k, y_1,\cdots,y_k$为$U(X)$中的元素且$\sum_{i=1}^{k}\|x_i-y_i\|\geqslant\varepsilon$时，有$\|x_1+\cdots+x_k+y_1+\cdots+y_k\|\leqslant 2k(1-\delta(\varepsilon))$. X叫做k一致光滑的，如果当$\tau\to 0$时，$\rho_{k,X}(\tau)/\tau\to 0$，其中$\rho_{k,X}(\tau)$规定为

$$2k\rho_{k,X}(\tau)=\sup\{\sum_{i=1}^{k}(\|x+\tau y_i\|+\|x-\tau y_i\|)-2k: \|x\|=\|y_i\|=1\ \ i=1,\cdots k,\}$$

本文证明上述k一致凸性等价于一致凸性，并且X为k一致光滑的当且仅当X为一致光滑的，因此这两个概念都不是新的概念.