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Sufficient Conditions of a Chaotic Map with Topological Entropy 0

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Let $f \in c^0(\mathrm{I})$ (where $\mathrm{I} = [0,1]$) and $\mathrm{ent}(f) = 0$, $x \in \overline{p(f)} - p(f)$. In this paper, through study of limit behavior of the sequence $\{f^{2^n}(x)\}_{n=0}^{\infty}$, we give two sufficient conditions of chaotic map with topological entropy 0, when the set of periodic points of f is not closed.

Lemma 1 [1] Let $f \in c^0(\mathrm{I})$. Topological entropy of f is zero if and only if the following conditions hold:

(1) For any positive integers m and k, $\forall x \in \mathrm{I}$, the sequence $\{f^{m\cdot 2^n+k}(x)\}_{n=0}^{\infty}$ has at most two limit points.

(2) If $x \in \overline{p(f)}$ and sequence $\{f^{m\cdot 2^n+k}(x)\}_{n=0}^{\infty}$ has two limit points, then they belong two endpoints of same connected component of $\mathrm{I} - \overline{p(f)}$, furthermore x is one of them.

Lemma 2 Let $f \in c^0(\mathrm{I})$. If the periods of periodic points of f are powers of 2, then

(i) If p is a periodic point of f with period 2^n, then $O(p, f^{2^i})$ is strongly separated under f^{2^i} for each $i = 0, 1, 2, \cdots, n-1$ ([2]).

(ii) If $x \in \Omega(f) - p(f)$, then $O(x, f^n)'$ is strongly separated under f^n for each $n > 0$. ([3]).

Lemma 3 [4] Let $f \in c^0(\mathrm{I})$ and the periods of periodic points of f be powers of 2. If $O(x, f^{kn})'$ is strongly separated under f^{kn} for some $n > 0$ and for each $k = 1, 2$, then the convex hull of $O(x, f^{2n})'$ contains no fixed point of f^n.

Lemma 4 [5] Let $f \in c^0(\mathrm{I})$. If there exist sequences $p_i \to x$ and $q_i \to y$ of periodic points, satisfying following conditions:

(1) $p_1 < p_2 < \cdots < p_i < \cdots < x < y < \cdots < q_i < \cdots < q_2 < q_1$;

(2) Either $x \bar{\in} p(f)$ or $y \bar{\in} p(f)$;

(3) There exists a sequence of positive integers $n_1, n_2, \cdots$, sach that for each $i > 0$,

$$f^{n_i}(J_i) \supset J_{i+1} \cup K_{i+1}, \quad f^{n_i}(K_i) \supset J_{i+1} \cup K_{i+1},$$

where $J_i = [p_i, P_{i+1}]$, $K_i = [q_{i+1}, q_i]$. then f is chaotic.

Lemma 5 Let $f \in c^0(\mathrm{I})$ and $\mathrm{ent}(f) = 0$, then the set of recurrent points is

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not closed if and only if there exsist $x \in \overline{p(f)} - p(f)$ sach that x is not a limit point of sequence $\{f^{2^n}(x)\}_{n=0}^{\infty}$.

Proof Necessity is clear. Hence we only prove sufficiency. If x is not a limit point of the sequence $\{f^{2^n}(x)\}_{n=0}^{\infty}$, by Lemma 1, the sequence $\{f^{2^n}(x)\}_{n=0}^{\infty}$ only has a limit point y. Without loss of generality we may assume that $x<y$.

Since $x \in \overline{p(f)} - p(f)$, it follows that $x \in \Omega(f) - p(f)$. By Lemma 2 and continuity of f, for each $n>0$, such that $f^{2^n}(x) \in \overline{p(f)} - p(f)$. Since $(x, y) \cap p(f) = \phi$(see [7]) thus $(x, y) \cap O(x, f) = \phi$.

Note that the sequence $\{f^{2^n}(x)\}_{n=0}^{\infty} \to y$, hence we can asseme that there exists some positive integer N such that for each nonegative integers i

$$f^{2^i}(x)<x, \text{ if } i \leqslant N; \quad f^{2^i}(x)>y>x, \text{ if } i>N.$$

By (ii) of Lemma 2 and Lemma 3, clearly, for each $i \leqslant N$ and each positive integers K, it follows that $f^{2^i(2K-1)}(x)<z_N<x$, where z_N is a fixed point of f.

Similarly, for each $i>N$ and each K, it follows that $f^{2^i(2K-1)}(x)>y>x$. Obviously, $O(x, f) - \{x\} = \bigcup_{\substack{i=0 \\ K=1}} \{f^{2^i(2K-1)}(x)\}$, hence $(z_N, y) \cap O(x, f) = \{x\}$.

Thus x is not a recurent point of f. Since $\overline{p(f)} = \overline{R(f)}$ hence $x \in \overline{R(f)} - R(f)$.

Theorem 1 Let $f \in c^0(\mathrm{I})$ and $\mathrm{ent}(f)=0$. If there exist $x \in \overline{p(f)} - p(f)$ sach that x is not a limit point of sequence $\{f^{2^n}(x)\}_{n=0}^{\infty}$ then f is chaotic.

Proof By Lemma 5 and Theorem A in [6], it is each easy to see that f is chaotic.

Theorem 2 Let $f \in c^0(\mathrm{I})$ and $\mathrm{ent}(f)=0$. If there exist $x \in \overline{p(f)} - p(f)$ sach that sequence $\{f^{2^n}(x)\}_{n=0}^{\infty}$ has two limit points, then f is chaotic.

Proof By Lemma 1, if sequence $\{f^{2^n}(x)\}_{n=0}^{\infty}$ has two limit points, then x is one of them.

Without loss of generality we may asseme that another limit point is y and $y>x$.

Since $y \in R(f) - p(f)$ (see [3]), thus $[x, y] \cap p(f) = \phi$. hence there exists a sequsec $n_i \to \infty$ of positive interges sach that $f^{2^{n_i}}(x)<x$, $f^{2^{n_i}}(x) \to x$ and $f^{2^{n_i}-1}(x) >y$, $f^{2^{n_i}-1}(x) \to y$. By (ii) of Lemma 2 and Lemma 3, $O(x, f^{2^{n_i}})$ is strongly separated under $f^{2^{n_i}}$ for each i and the convex hall of $O(x, f^{2^{n_i}})$ contains no fixed point of $f^{2^{n_i}-1}$. Hence we can select a sequence $p_i \to x$ and sequence $n_i \to \infty$ of positive integers, such that for each $i=1, 2, \cdots$, satisfying following conditions:

(1) p_i is a periodic point of f with period 2^{n_i};

(2) $p_i<p_{i+1}<x$ and $n_i<n_{i+1}$;

(3) (p_i, x) contains no fixed point of $f^{2^{n_i}}$.

For each $i>0$, let $k_i = 2^{n_i - 1}$ and $q = f^{K_i}(p_i)$.

Since $f^{k_i}(x) > x$ and $[x, y] \cap p(f) = \phi$, condition (3) implies $y < q_i$.

On the one hand, since $O(p_1, f^{k_1})$ is strongly separated under f^{k_1}, there exists a fixed point z_1 of f^{k_1} with $z_1 \in (p_1, q_1)$.

On the other hand, if K is the convex hull of $O(p_2, f^{K_2})$, then by (i) of Lemma 2 and Lemma 3, K is contians not fixed point of $f^{K_2/2}$. Since $n_1 < n_2$, $K_2/2$ is divisible by K_1. It follows that $f^{K_2/2}(z_1) = z_1$, since $f^{K_1}(z_1) = z_1$ Thus $z_1 \notin K$. This implies that $y < q_2 < z_1 < q_1$. Furthermore, using induction on i, we have

$$p_1 < p_2 < \cdots < p_i < \cdots < x < y < \cdots < q_i < \cdots < q_2 < q_1$$

Note that $f^{K_i}(p_i) = q_i$ and $f^{K_i}(q_i) = p_i$. It is easy to check that the sequences p_i and q_i satisfy the hypothesis of Lemma 4. Hence by Lemma 4, f is chaotic.

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拓扑熵为零的混乱映射的充分条件

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摘要 线段上的连续自映射，当周期点集为闭集时，其轨道十分简单，当然，动力系统不会是混乱的，因此，研究周期点集的聚点的极限性态与混乱的关系，无疑可以进一步揭示混乱现象产生的原因。文〔6〕证明了当回归点集非闭，f 是混乱的。本文则给出了周期点集非闭时 f 为混乱的充分条件。这说明了只要周期点集非闭动力系统就可能是混乱的。

设 $f \in c^0(\mathrm{I})$ 和 $\mathrm{ent}(f) = 0, x \in \overline{p(f)} - p(f)$.本文通过研究 x 轨道上的点列 $\{f^{2^n}(x)\}_{n=0}^{\infty}$ 的极限性态，给出了当 f 的周期点集非闭时，f 为混乱的两个充分条件。

定理1 设 $f \in c^0(\mathrm{I})$ 和 $\mathrm{ent}(f) = 0$, 如果存在 $x \in \overline{p(f)} - p(f)$ 使 x 不为 $\{f^{2^n}(x)\}_{n=0}^{\infty}$ 的极限点时，则 f 是混乱的。

定理2 设 $f \in c^0(\mathrm{I})$ 和 $\mathrm{ent}(f) = 0$, 如果存在 $x \in \overline{p(f)} - p(f)$ 使 $\{f^{2^n}(x)\}_{n=0}^{\infty}$ 有两个极限点时，则 f 是混乱的。