Journal of Mathematical Research and Exposition
Vol.10 No.1 Feb. 1990

Composition Operator on Hardy Space*

Tong Yusun

(Department of Mathematics, Fudan University, Shanghai)

For composition operators defined on Hardy space with respect to the unit disk, extensive results have been obtained. In this paper, we study composition operators on $H^p(B)$ $(1<p<\infty)$, where B is the unit ball in the n-dimensional space $\mathbf{C}^n$.

In what follows, the element $(z_1, z_2, \cdots, z_n)$ of $\mathbf{C}^n$ will be denoted by z, and $z^m=(z_1^{m_1}, z_2^{m_2}, \cdots, z_n^{m_n})$ for any ordered n-tuple $m=(m_1, m_2, \cdots, m_n)$ of nonnegative integers. For convenience, the same notation will be used to represent the monomials of several variables $z=z_1z_2\cdots z_n$, $z^m=z_1^{m_1}z_2^{m_2}\cdots z_n^{m_n}$. Doing so will cause no confusion and the reader is able to tell the meaning of the notation from the context.

Suppose φ is a mapping $B\to B$, the composition operator C_φ is defined by $C_\varphi f=f\circ\varphi$. First, we give a characterization of this class of operators.

Theorem I Let T be a bounded linear operator on $H^p(B)$ $(1<p<\infty)$, and Tz be not a constant. Then T is a composition operator if and only if

$$Tz^m=(Tz)^m. \tag{1}$$

holds for any ordered n-tuple m of nonnegative integers.

Proof Necessity is evident. We only prove sufficiency. For any $z_0\in B$, consider the map Γ from $H^p(B)$ into $\mathbf{C}$: $f\mapsto(Tf)(z_0)$. Put $\pi_i(z)=z_i$, then $\pi_i\in H^p(B)$. Define $\varphi_i(z_0)=\Gamma\pi_i$, we have

$$(T\pi_i)(z_0)=\Gamma\pi_i=\varphi_i(z_0). \tag{2}$$

Define $\varphi(z)=(\varphi_1(z), \varphi_2(z), \cdots, \varphi_n(z))$. From (2), it follows that

$$(T\pi_i)(z)=\pi_i\varphi(z). \tag{3}$$

It follows from the definition that φ is a holomorphic map and for any polynomial p defined on B.

$$(Tp)(z)=p(\varphi(z)). \tag{4}$$

holds.

To prove (4) hold not only for any polynomial p but also for any $f\in H^p(B)$, we should prove $\varphi(z)\in B$, where $z\in B$.

* Received Apr. 26, 1988.

Consider the ball algebra $A(B)$, which is the class of all $f:B\to\mathbf{C}$ that are continuous on $\bar{B}$ and holomorphic in B. Equipped with the supremum norm, $A(B)$ is a Banach algebra. Obviously, $A(B)\subset H^p(B)$. Take a linear functional Γ' on $A(B)$:

$$\Gamma'=\Gamma|A(B).$$

From $Tz^m=(Tz)^m$ and the definition of Γ, we have

$$\begin{aligned}\Gamma'(p_1p_2)&=\Gamma(p_1p_2)=T(p_1p_2)(z_0)\\&=(Tp_1)(z_0)(Tp_2)(z_0)=\Gamma(p_1)\Gamma(p_2)=\Gamma'(p_1)\Gamma'(p_2)\end{aligned}$$

holds for any polynomials p_1, p_2. Thus it is easy to see that Γ' is a linear multiplicative functional on $A(B)$, so $\|\Gamma'\|=1$.

Now, pick θ_i such that $|\varphi_i(z_0)|^2=e^{i\theta_i}\varphi_i(z_0)^2$. Take a polynomial $p(z)=e^{i\theta_1}z_1^2+\cdots+e^{i\theta_n}z_n^2$. We have

$$|\varphi(z_0)|^2=p(\varphi(z_0))=(Tp)(z_0)=\Gamma(p)=\Gamma'(p)\leqslant\|p\|=\max_{z\in\bar{B}}|p(z)|\leqslant 1.$$

Therefore $\varphi(B)\subset\bar{B}$. Suppose there exists $z_0\in B$ such that $|\varphi(z_0)|=1$. From maximum modulus theorem for holomorphic function,

$$\langle\varphi(z),\varphi(z_0)\rangle\equiv C,\quad z\in B. \tag{5}$$

holds, where C is a constant with modulo 1. Thus $\varphi(z)=C\varphi(z_0)$. Take $p(z)=z_1z_2\cdots z_n$. It follows from (4) that

$$(Tp)(z)=p(\varphi(z))=\varphi_1(z)\cdots\varphi_n(z)=\text{const}$$

which contradicts that Tp is not a constant. Hence $\varphi(B)\subset B$.

Using (4), boundedness of T, and density of polynomials in $H^p(B)$, we conclude that $(Tf)(z)=f(\varphi(z))$ holds for any $f\in H^p(B)$, which means that $T=C_\varphi$. QED.

Denote Moebius group on B by $\mathrm{Aut}(B)$. From Theorem 1, we can get a necessary and sufficient condition that a composition operator is invertible, which coincides with the case of unit disk.

Theorem 2 Let C_φ be a composition operator on $H^p(B)$. Then C_φ is invertible if and only if $\varphi\in\mathrm{Aut}(B)$.

Proof Suppose that $\varphi\in\mathrm{Aut}(B)$. It is easily seen that $C_\varphi^{-1}=C_{\varphi^{-1}}$. Conversely, suppose that C_φ is invertible. Denote $T=C_\varphi^{-1}$. Then for ordered n-tuples of nonnegative integers, l and m, we have

$$\begin{aligned}(Tz^{l+m})\circ\varphi&=C_\varphi Tz^{l+m}=z^{l+m}=z^lz^m\\&=(Tz^l\circ\varphi)(Tz^m\circ\varphi)=(Tz^lTz^m)\circ\varphi\end{aligned}$$

which implies

$$(Tz^{l+m}-Tz^lTz^m)\circ\varphi=0. \tag{6}$$

Since C_φ is invertible, the map φ should be not a constant. Thus the range of φ is an open set in B. It follows from (6) that $Tz^{l+m}=Tz^lTz^m$. Particularly, $Tz^m=$

$(Tz)^m$. In view of $C_\varphi Tz=z$, so Tz is not a constant. Now, it follows from Theorem 1 that T is a composition operator. Suppose $T=C_\psi$, i.e. $C_\psi=C_\varphi^{-1}$. We have $\varphi(\psi(z))=\psi(\varphi(z))=z$, therefore $\varphi\in\mathrm{Aut}(B)$. QED.

To give an estimation of the norm of composition operator, we should do some preparations. Suppose that $f\in H^p(B)$ $(1\leqslant p<\infty)$. Denote $f_r(z)=f(rz)$ for $0\leqslant r<1$, and

$$I_p=\sup_{0<r<1}\left\{\int_{\partial B}|f_r|^p\mathrm{d}\sigma\right\}^{\frac{1}{p}}<+\infty, \tag{7}$$

where σ is rotation-invariant positive Borel measure on ∂B for which $\sigma(\partial B)=1$. Put

$$P(z,\zeta)=\frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}},\quad z\in B,\ \zeta\in\partial B.$$

From the knowledge of [3] chapter 5, it follows that in case $p=1$, there exists a least M-harmonic majorant u in B, $|f|\leqslant u$, and a positive measure μ on ∂B such that

$$u=P[\mu],\quad I_1=\|\mu\|, \tag{8}$$

where $P[\mu](z)=\int_{\partial B}P(z,\zeta)\mathrm{d}\mu(\zeta)$; in case $1<p<+\infty$, there exists a least M-harmonic majorant u in B, $|f|^p\leqslant u$, and $f^*\in L^p(\sigma)$, such that

$$u=P[|f^*|^p],\quad I_p=\|f^*\|, \tag{9}$$

where $P[|f^*|^p](z)=\int_{\partial B}P(z,\zeta)|f^*(\zeta)|^p\mathrm{d}\sigma_\zeta$. With the above symbols and notations, we have the following lemma and theorem.

Lemma 1
$$(I_p)^p=u(0). \tag{10}$$

Proof If $p=1$, from [3] Theorem 3.3.4, $\lim\limits_{r\nearrow 1}u_r\mathrm{d}\sigma=\mathrm{d}\mu$.

in the weak*-topology of the dual space of $C(\partial B)$, where $u_r(z)=u(rz)$, for all $z\in\partial B$, and μ is a positive measure on ∂B satisfying (8). Thus $\lim\limits_{r\nearrow 1}\int_{\partial B}u_r\mathrm{d}\sigma=\int_{\partial B}\mathrm{d}\mu=\|\mu\|$. Since $\int_{\partial B}u_r\mathrm{d}\sigma=u(0)$, using (8) we obtain $I_1=u(0)$.

If $1<p<\infty$, from [3] Theorem 3.3.4, $\lim\limits_{r\nearrow 1}\|u_r-|f^*|^p\|_1=0$, so that $\lim\|u_r\|_1=\|f^*\|_p^p$. But $\|u_r\|_1=u(0)$, by (9) $u(0)=(I_p)^p$. QED.

Theorem 3 Let $\varphi:B\to B$ be a holomorphic map. Then for composition operator $C_\varphi:H^p(B)\to H^p(B)$ $(1\leqslant p<\infty)$, the following estimate

$$\|C_\varphi\|\leqslant\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{n}{p}} \tag{11}$$

holds.

Proof (i) If $p=1$, $f\in H^1(B)$, then take least M-harmonic majorants $u_f, u_{f\circ\varphi}$ and a positive measure μ_f on ∂B such that

$$|f(z)|\leqslant u_f(z)=P[\mu_f](z),\quad |f\circ\varphi(z)|\leqslant u_{f\circ\varphi}(z),\quad \|f\|=\|\mu_f\|. \tag{12}$$

By the definition,

$$u_{f\circ\varphi}(z)\leqslant u_f(\varphi(z)). \tag{13}$$

It follows from Lemma 1 and (8), (12), (13) that

$$\|C_\varphi f\| = \|f\circ\varphi\| = u_{f\circ\varphi}(0)\leqslant u_f(\varphi(0))$$
$$= P[\mu_f](\varphi(0)) = \int_{\partial B}\frac{(1-|\varphi(0)|^2)^n}{|1-\langle\varphi(0),\zeta\rangle|^{2n}}\mathrm{d}\mu_f(\zeta)$$
$$\leqslant\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^n\int_{\partial B}\mathrm{d}\mu_f(\zeta) = \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^n\|f\|.$$

(ii) If $1<p<\infty$, similar to (i), using Lemma 1 and the illustration before this lemma, we have

$$\|C_\varphi f\|^p = u_{f\circ\varphi}(0)\leqslant u_f(\varphi(0))$$
$$= \int_{\partial B}P(\varphi(0),\zeta)\,|f^*(\zeta)|^p\mathrm{d}\sigma_\zeta\geqslant\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^n\|f\|^p. \quad \text{QED}.$$

Corollary 1 Let $F:B\to B$ be a holomorphic map, $F(a)=0$, and $\varphi\in\mathrm{Aut}(B)$, $\varphi(a)=0$. Then for composition operator $C_F:H^p(B)\to H^p(B)$ $(1\leqslant p<\infty)$,

$$\|C_F\|^p\leqslant\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^n. \tag{14}$$

Proof It is evident that $F\circ\varphi^{-1}:B\to B$ is a holomorphic map, and $F\circ\varphi^{-1}(0)=0$. [3] Theorem 8.1.2 implies

$$|F\circ\varphi^{-1}(z)|\leqslant|z|.$$

for all $z\in B$. Set $z=\varphi(0)$. It follows that $|F(0)|\leqslant|\varphi(0)|$. Applying Theorem 3 and the monotonicity of $\frac{1+x}{1-x}$ for $0\leqslant x<1$, we have

$$\|C_F\|^p\leqslant\left(\frac{1+|F(0)|}{1-|F(0)|}\right)^n\leqslant\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^n. \quad \text{QED}.$$

Suppose $\zeta\in\partial B$, $a>1$. **Denote** $D_a(\zeta)=\{z\,\big|\,|1-\langle z,\zeta\rangle|<\frac{a}{2}(1-|z|^2)\}$. **If for every** $a>1$ **and for every sequence** $\{z_i\}$ **in** $D_a(\zeta)$ **that converges to** ζ, $f(z_i)$ **converges** to a same point, then we denote this common limit by k-$\lim f(\zeta)$ ([3], 5.4, 6). From [3] Theorem 5.6.4, if $f\in H^p(B)$, then $f^*(\zeta)=k$-$\lim f(\zeta)$ hold for almost all $\zeta\in\partial B$, where f^* is the function defined on ∂B, which appeared at the illustration before Lemma 1.

Lemma 2 Let $f\in H^p(B)$ $(1\leqslant p<\infty)$, $\varphi\in\mathrm{Aut}(B)$. Then $f^*\circ\varphi=(f\circ\varphi)^*$, a.e. on ∂B.

Proof From [3] Theorem 5.6.4, there exists a subset A of ∂B with measure σ zero, such that $f^*(\xi)=k$-$\lim f(\xi)$ exist for $\xi\overline{\in}A$. By χ_S we denote the characteristic function of set S. [3] Theorem 3.3.8 yields that

$$\int_{\partial B}\chi_{\varphi^{-1}(A)}(z)\mathrm{d}\sigma_z=\int_{\partial B}P(\varphi(0),\zeta)\chi_{\varphi^{-1}(A)}(\varphi^{-1}(\zeta))\mathrm{d}\sigma_\zeta$$
$$=\int_{\partial B}P(\varphi(0),\zeta)\chi_A(\zeta)\mathrm{d}\sigma_\zeta$$
$$\leqslant\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^n\int_{\partial B}\chi_A(\zeta)\mathrm{d}\sigma_\zeta=0.$$

i.e. $\varphi^{-1}(A)$ is also a subset of ∂B with measure zero. Since $\varphi \in \mathrm{Aut}(B)$, by [3] Theorem 2.2.2 and Theorem 2.2.5, we can choose $a \in \partial B$ such that

$$\frac{1-|\varphi(z)|^2}{1-|z|^2}=\frac{1-|a|^2}{|1-\langle z,a\rangle|^2},$$

so that

$$\frac{1-|a|}{1+|a|}\leqslant\frac{1-|\varphi(z)|^2}{1-|z|^2}\leqslant\frac{1+|a|}{1-|a|}.$$

Besides, for any $z_i \in B$, [3] Theorem 8.1.4 (a generalization of Schwarz lemma) implies

$$\frac{|1-\langle\varphi(z),\varphi(z_i)\rangle|^2}{(1-|\varphi(z)|^2)^2}\leqslant\frac{1-|\varphi(z_i)|^2}{1-|z_i|^2}\cdot\frac{|1-\langle z,z_i\rangle|^2}{(1-|z|^2)^2}\cdot\frac{1-|z|^2}{1-|\varphi(z)|^2}$$

$$\leqslant\left(\frac{1+|a|}{1-|a|}\right)^2\frac{|1-\langle z,z_i\rangle|^2}{(1-|z|^2)^2}$$

Let $z_i \to \zeta \overline{\in} \varphi^{-1}(A)$, we have

$$\frac{|1-\langle\varphi(z),\varphi(\zeta)\rangle|^2}{(1-|\varphi(z)|^2)^2}\leqslant\left(\frac{1+|a|}{1-|a|}\right)^2\frac{|1-\langle z,\zeta\rangle|}{(1-|z|^2)^2}$$

which means $\varphi(D_\alpha(\zeta))\subset D_{\frac{1+|a|}{1-|a|}\alpha}(\varphi(\zeta))$. Since $\varphi(z)$ tends $\varphi(\zeta)$ as $z\to\zeta$,

$$(f\circ\varphi)^*(\zeta)=k\text{-}\lim\,(f\circ\varphi)(\zeta)=\lim_{\substack{z\to\zeta\\ z\in D_\alpha(\zeta)}}f(\varphi(z))=k\text{-}\lim f(\varphi(\zeta))=f^*(\varphi(\zeta)).$$

QED.

Corollary 2 Let $\varphi \in \mathrm{Aut}(B)$, $a=\min(|\varphi(0)|,\ |\varphi^{-1}(0)|)$. Then $\|C_\varphi\|^p\geqslant\left(\frac{1-a}{1+a}\right)^n$.

Proof First, since $\varphi^{-1}\in\mathrm{Aut}(B)$, it follows from Theorem 3 that

$$\|C_\varphi\|^p\geqslant\|C_{\varphi^{-1}}\|^{-p}\geqslant\left(\frac{1-|\varphi^{-1}(0)|}{1+|\varphi^{-1}(0)|}\right)^n.$$

Secondly, from Lemma 2, [3] Theorem 5.6.8 (a) and Theorem 3.3.8,

$$\|C_\varphi f\|^p=\int_{\partial B}|(f\circ\varphi)^*(\zeta)|^p d\sigma_\zeta=\int_{\partial B}|f^*(\varphi(\zeta))|^p d\sigma_\zeta$$

$$=\int_{\partial B}p(\varphi(0),z)\,|f^*(z)|^p d\sigma_z\geqslant\left(\frac{1-|\varphi(0)|}{1+|\varphi(0)|}\right)^n\|f\|^p.\quad \text{QED.}$$

References

[1] Nordgren, E, A., Canad. J. Math. 20(1968), 442—449.

[2] Nordgren, E. A., Composition operators on Hilbert space, in Hilbert space operators, Lecture Notes in Math. 693, Springer-Verlag, Berlin, 1978, 37—53.

[3] Rudin, W., Function Theory in the Unit Ball of C^n, Springer-Verlag, 1980.

Hardy 空间上的复合算子

童　裕　孙

（复旦大学数学系，上海）

摘要　本文讨论 n 维复单位球的Hardy空间上的复合算子，主要是利用Banach代数和n维单位球上解析函数的理论，给出了这类空间上复合算子的一个特征，证明了复合算子可逆的充要条件是其符号函数属于单位球的自同构群，并且对复合算子的范数作出了估计.