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# On the Existence of Borel Point of Meromorphic Algebroidal Functions in Unit Circle

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**Abstract**: In this paper, the author defines Borel point for $\nu$-valeus meromorphic algebroidal functions of positive finite order in unit circle, and proves existence theorem of it.

## I. Introduction

About Borel direction of meromorphic algebroidal functions, Lü and Gu[1] proved its existence. As for meromorphic functions, Sun[2] proved the existence of Nevanlinna point. This paper is based on their ideas. It obtanned the result of the abstract.

$\nu$-value meromorphic algebroidal functions $W=W(z)$ in unit circle is defin ned by function equation:

$$A_\nu(z)W^\nu + A_{\nu-1}(z)W^{\nu-1} + \cdots + A_0(z) = 0 \qquad (1)$$

Here $A_\nu(z), \cdots, A_0(z)$ are entive function in unit circle. They do not have common zero points. $W(z)$ is a $\nu$-value function $(\nu>1)$. [When $\nu=1$, $W(z)$ is a meromorphic function]. It's domain of simple value is a Riemann surface $\widetilde{R}_z$. $\widetilde{R}_z$ is $\nu$-sheeted covering surface of unit circle. For any point set $E\subset\{|z|<1\}$ or $|z|<r$ $(0<r<1)$. The part in $\widetilde{R}_z$ corresponding to $E$ or $|z|<r$ is denoted b by $\widetilde{E}$ or $|\widetilde{z}|<r$. In this paper: $0<r<1$. By

$$\Delta(\varphi_1,\varphi_2,1) = \{z \mid \varphi_1<\arg z<\varphi_2,\ |z|<1\}$$
$$\Delta(\varphi_1,\varphi_2;1) = \{z \mid \varphi_1\leqslant\arg z\leqslant\varphi_2,\ |z|<1\}$$

$(0\leqslant\varphi_1<\varphi_2<2\pi)$, we denote the domain of circular sector in unit circle, then let

$$S(r) = \frac{1}{\pi}\iint_{|\widetilde{z}|<r}\left(\frac{|W'(z)|}{1+|W(z)|^2}\right)^2 d\Omega,\quad T(r) = \frac{1}{\nu}\int_0^r \frac{S(r)}{r}dr.$$

By $n(r,a)$ and $n(r,\Delta(\varphi_1,\varphi_2;1),a)$, we denote the numbers of zeros of function $W(z)-a$ in $|z|<r$ and $\widetilde{\Delta}(\varphi_1,\varphi_2;1)\cap\{|z|<r\}$. Including order of zero points, let

$$N(r,a) = \frac{1}{\nu}\int_0^r \frac{n(r,a)-n(0,a)}{r}dr + \frac{n(0,a)}{\nu}\log r.$$

At last, by $n(r,\widetilde{R}_z)$ and $n(r,\ \Delta(\varphi_1,\varphi_2;1),\widetilde{R}\ )$, we denote the numbers of the branch points of $Rz$ in $|z|\leqslant r$ and $\widetilde{\widetilde{\Delta}}(\varphi_1,\varphi_2;1)\cap\{|\widetilde{z}|\leqslant r\}$, Including order of the branch points, let

$$N(r,\widetilde{R}_z)=\frac{1}{v}\int_0^r\frac{n(r,\widetilde{R}_z)-n(0,\widetilde{R}_z)}{r}dr+\frac{n(0,\widetilde{R}_z)}{v}\log r,$$

there, we have

$$N(r,\widetilde{R}_z)\leqslant 2(v-1)T(r)+O(1). \tag{2}$$

**Definition** Let $W=W(z)$ be a $v$-value meromorphic algerbroidal function defined by (1) in unit circle. It's positive finite order:

$$\rho=\overline{\lim_{r\to 1}}\frac{\log^+T(r)}{\log\frac{1}{1-r}}\qquad(0<\rho<+\infty).$$

Let exist $\varphi$, such that

$$\overline{\lim_{r\to 1}}\frac{\log^+n(r,\ \Delta(\varphi-\delta,\varphi+\delta;1),a)}{\log\frac{1}{1-r}}=\rho\ .$$

At most, it has $2v$ numbers of exceptional value $a$, for any given $\delta(0<\delta<\frac{\pi}{2})$ and complex value $a$. Then, we called $e^{i\varphi}$ a Borel point of $v$-value meromorphic algebroidal function $W(z)$ of positive finite order $\rho$.

Based on the theory of algebroidal function, we called "$a$" the Borel exceptional value of $W(z)$ in domain of circular sector $\triangle(\varphi-\delta,\varphi+\delta;1)$, the value "$a$" determines the inequallity

$$\overline{\lim_{r\to 1}}\frac{\log^+n(r,\ \Delta(\varphi-\delta,\varphi+\delta;1),a)}{\log\frac{1}{1-r}}<\rho$$

$(0<\rho<\infty)$, $2v$ numbers of exceptional value $a$ of dafinition are all Borel exceptional value.

## 2. Several Lemmas

**Lemma 1** Let $W=W(z)$ be a $v$-value algebroidal function in unit circle fixed by (1). $a_1,\cdots,a_q$ $(q>2)$ are points on the $W$-sphere, and

$$\sum_{j=1}^{q}n(1,a_j)<\infty\quad\text{and}\quad n(1,\widetilde{R}_z)<\infty,$$

then

$$(q-2)S(r)\leqslant\sum_{j=1}^{q}n(1,a_j)+n(1,\widetilde{R}_z)+\frac{A}{1-k}\quad(0<k<1).$$

Where $A$ is defined by $a_1,\cdots,a_q$.

Proof in [3]

**Lemma 2** Let $W=W(z)$ be a $v$-value meromorphic algebroidal function in unit circle fixed by (1).

Then, as for $0<\delta<\delta_0<\frac{\pi}{2}$ and $0\leqslant\theta_0<2\pi$, let

$$\Delta_0=\{z \mid |\arg z-\theta_0|<\delta_0,\ |z|<1\}$$
$$\overline{\Delta}_0=\{z \mid |\arg z-\theta_0|\leqslant\delta,\ |z|<1\}$$
$$S(r,\overline{\Delta}_0)=\frac{1}{\pi}\iint_{\widetilde{\Delta}_0\cap(|z|<r)}\left(\frac{|W'(z)|}{1+|W(z)|^2}\right)^2 d\Omega$$

then, any time there is

$$(q-2)S(r,\overline{\Delta}_0)\leqslant\sum_{j=1}^{q}n(r^\lambda,\Delta_0,a_j)+n(r^\lambda,\Delta_0,\widetilde{R}_z)+O(\log\frac{1}{1-r})$$

for any given number $\lambda$ $(0<\lambda<1)$, and the points $a_1,\cdots,a_q(q>2)$ on the $W$-sphere.

**Proof** Let $r_n=(0,1)^{\lambda^{n'}}$ $(n=0,1,2,\cdots)$, then, let $R=r_{n+1}$, we can see the function

$$\zeta=f(z)=\frac{(2e^{-i\theta_0})^{\frac{\pi}{\delta_0}}+2(ze^{-i\theta_0})^{\frac{\pi}{2\delta_0}}R^{\frac{\pi}{2\delta_0}}-R^{\frac{\pi}{\delta_0}}}{(ze^{-i\theta_0})^{\frac{\pi}{\delta_0}}-2(ze^{-i\theta_0})^{\frac{\pi}{2\delta_0}}R^{\frac{\pi}{2\delta_0}}-R^{\frac{\pi}{\delta_0}}} \qquad (3)$$

it map the domain circular sector $E=\Delta_0\cap\{|z|\leqslant R\}$ onto $|\zeta|<1$, it map the point $z=(\sqrt{2}-1)^{\frac{2\delta_0}{\pi}}Re^{i\theta_0}$ onto center of a circle $\zeta=0$; samely, it map $F=\widetilde{\Delta}_0\cap\{r_1\leqslant|z|\leqslant r_n\}$ into $|\zeta|<1$, $F$ belong to $E$.

Then, we will work about the $\max\limits_{z\in F}\{\frac{1}{1-|\zeta|}\}$ by (3). Let $z=he^{i(\theta_0+\varphi)}\in F$ $(|\varphi|\leqslant\delta, r_1\leqslant h\leqslant r_n)$, then, from (3), and let $f(z)=\frac{A+iB}{C+iD}$, we obtain

$$A=h^{\frac{\pi}{\delta_0}}\cos\frac{\varphi\pi}{\delta_0}+2R^{\frac{\pi}{2\delta_0}}h^{\frac{\pi}{2\delta_0}}\cos\frac{\varphi\pi}{2\delta_0}-R^{\frac{\pi}{\delta_0}};\quad B=h^{\frac{\pi}{\delta_0}}\sin\frac{\varphi\pi}{\delta_0}+2R^{\frac{\pi}{2\delta_0}}h^{\frac{\pi}{2\delta_0}}\sin\frac{\varphi\pi}{2\delta_0};$$
$$C=h^{\frac{\pi}{\delta_0}}\cos\frac{\varphi\pi}{\delta_0}-2R^{\frac{\pi}{2\delta_0}}h^{\frac{\pi}{2\delta_0}}\cos\frac{\varphi\pi}{2\delta_0}-R^{\frac{\pi}{\delta_0}};\quad D=h^{\frac{\pi}{\delta_0}}\sin\frac{\varphi\pi}{\delta_0}-2R^{\frac{\pi}{2\delta_0}}h^{\frac{\pi}{2\delta_0}}\sin\frac{\varphi\pi}{2\delta_0}$$
$$A^2+B^2=P-Q(R^{\frac{\pi}{\delta_0}}-h^{\frac{\pi}{\delta_0}})-G;\quad C^2+D^2=P+Q(R^{\frac{\pi}{\delta_0}}-h^{\frac{\pi}{\delta_0}})-G.$$

In it

$$P=4R^{\frac{\pi}{\delta_0}}h^{\frac{\pi}{\delta_0}}+R^{\frac{2\pi}{\delta_0}}+h^{\frac{2\pi}{\delta_0}};\quad Q=4R^{\frac{\pi}{2\delta_0}}h^{\frac{\pi}{2\delta_0}}\cos\frac{\varphi\pi}{2\delta_0};\quad G=2R^{\frac{\pi}{\delta_0}}h^{\frac{\pi}{\delta_0}}\cos\frac{\varphi\pi}{\delta_0},$$

we can obtain

$$|\zeta|=|f(z)|=\sqrt{\frac{A^2+B^2}{C^2+D^2}}=\sqrt{1-\frac{2Q(R^{\frac{\pi}{\delta_0}}-h^{\frac{\pi}{\delta_0}})}{P+Q(R^{\frac{\pi}{\delta_0}}-h^{\frac{\pi}{\delta_0}})-G}}$$
$$\leqslant 1-\frac{Q(R^{\frac{\pi}{\delta_0}}-h^{\frac{\pi}{\delta_0}})}{P+Q(R^{\frac{\pi}{\delta_0}}-h^{\frac{\pi}{\delta_0}})-G}$$

From it, we obtain

$$\frac{1}{1-|\zeta|}<\frac{P+Q(R^{\frac{\pi}{\delta_0}}+h^{\frac{\pi}{\delta_0}})+G}{Qh^{\frac{\pi}{\delta_0}}((\frac{R}{h})^{\frac{\pi}{\delta_0}}-1)} \tag{4}$$

As

$R=r_{n+1}$, $(0,1)^{\lambda}=r_1<h<r_n$, $0<\delta<\delta_0<\frac{\pi}{2}$, $0<\cos\frac{\delta\pi}{2\delta_0}<\cos\frac{\varphi\pi}{2\delta_0}$, then obtain

$$\frac{P+Q(R^{\frac{\pi}{\delta_0}}+h^{\frac{\pi}{\delta_0}})+G}{Qh^{\frac{\pi}{\delta_0}}}<\frac{16R^{\frac{2\pi}{\delta_0}}}{4h^{\frac{2\pi}{\delta_0}}\cos\frac{\delta\pi}{2\delta_0}}<\frac{4r_{n+1}^{\frac{2\pi}{\delta_0}}}{r_1^{\frac{2\pi}{\delta_0}}\cos\frac{\delta\pi}{2\delta_0}}<\frac{4r_{n+1}^4}{(0.1)^{\frac{2\pi\lambda}{\delta_0}}\cos\frac{\delta\pi}{2\delta_0}}$$

$$(\frac{R}{h})^{\frac{\pi}{\delta_0}}-1\geqslant(\frac{r_{n+1}}{r_n})^{\frac{\pi}{\delta_0}}-1=(1+\frac{1-\frac{r_n}{r_{n+1}}}{\frac{r_n}{r_{n+1}}})^{\frac{\pi}{\delta_0}}-1$$

$$>\frac{\pi}{\delta_0}\frac{r_{n+1}}{r_n}(1-\frac{r_n}{r_{n+1}})>2\frac{r_{n+1}}{r_n}(1-\frac{r_n}{r_{n+1}}).$$

From (4) and the above estimates, we obtain $\frac{1}{1-|\zeta|}<N\frac{r_{n+1}^3 r_n}{1-\frac{r_n}{r_{n+1}}}$. In it

$N=\frac{2}{(0.1)^{\frac{2\pi\lambda}{\delta_0}}\cos\frac{\delta\pi}{2\delta_0}}$ is a positive constant. Where $r>r_1$, there exists $n>1$ anytime, let $r_n<r<r_{n+1}$, then, we obtain $R=r_{n+1}=(0.1)^{\lambda^{n+1}}=(0.1)^{\lambda^n\lambda}=r_n^{\lambda}<r^{\lambda}$, from it, we obtain

$$\frac{1}{1-|\zeta|}<N\frac{r^{3\lambda+1}}{1-r_n^{1-\lambda}}<N\frac{r^{3\lambda+1}}{1-r^{1-\lambda}}.$$

As $0<\lambda<1$, there is $3\lambda+1>1-\lambda$ (unless $\lambda<0$), then

$$\frac{1}{1-|\zeta|}<N\frac{r^{1-\lambda}}{1-r^{1-\lambda}}.$$

Then, there exist a value $r_0$ anytime ($r_0<1$), when $r_0<r<1$, from it, we have

$$\frac{1}{1-|\zeta|}<N\frac{r^{1-\lambda}}{1-r^{1-\lambda}}<N\log(1+\frac{r^{1-\lambda}}{1-r^{1-\lambda}})=N\log\frac{1}{1-r^{1-\lambda}}$$

Because $\lim\limits_{r\to1}\log\frac{1}{1-r^{1-\lambda}}/\log\frac{1}{1-r}=1$, then, we obtain $\frac{1}{1-|\zeta|}<O(\log\frac{1}{1-r})$.

Then, because $W=W(z)=W(f^{-1}(\zeta))$ has been defined in $|\zeta|<1$, let

$$S(r,F)=\frac{1}{\pi}\iint_{F}(\frac{|W'(z)|}{1+|W(z)|^2})^2 d\Omega$$

Then, from conforal constancy of covering area $S(r,F)$, as for $F\subset E$, from Lemma 1, we obtain

$$(q-2)S(r,F)<\sum_{j=1}^{q}n(E,a_j)+n(E,R_z)+O(\log\frac{1}{1-r})$$

having pay attention

$$S(r,F)=S(r,\overline{\Delta}_0)-S(r_1,\Delta_0)$$

$$n(E,a_j)=n(r_{n+1},\ \Delta_0,a_j)<n(r^\lambda,\ \Delta_0,a_j)$$
$$n(E,\widetilde{R}_z)=n(r_{n+1},\ \Delta_0,\ \widetilde{R}_z)<n(r^\lambda,\ \Delta_0,\widetilde{R}_z)$$

Then, we obtain

$$(q-2)S(r,\ \overline{\Delta}_0)<\sum_{j=1}^{q}n(r^\lambda,\ \Delta_0,a_j)+n(r^\lambda,\ \Delta_0,\widetilde{R}_z)$$
$$+(q-2)S((0.1)^\lambda,\ \Delta_0)+O(\log\frac{1}{1-r})$$

Put the given value $(q-2)S((0.1)^\lambda,\ \overline{\Delta}_0)$ into $O(\log\frac{1}{1-r})$, by proof, we obtain

$$(q-2)S(r,\ \overline{\Delta}_0)<\sum_{j=1}^{q}n(r^\lambda,\ \Delta_0,a_j)+n(r^\lambda,\Delta_0,\widetilde{R}_z)+O(\log\frac{1}{1-r})$$

$(0<r<1)$, then finish the proof of lemma 2.

**Lemma 3** Let $W=W(z)$ be a $\nu$-value meromorphic algebroidal function in unit circle fixed by (1), it's positive finite order:

$$\rho=\overline{\lim_{r\to1}}\frac{\log^+T(r)}{\log\frac{1}{1-r}}\qquad(0<\rho<+\infty),$$

let $m$ be a positive integer, let $\theta_0=0,\ \theta_1=\frac{2\pi}{m},\cdots,\ \theta_{m-1}=(m-1)\frac{2\pi}{m};\ \theta_m=\theta_0$. let domain of circular sector

$$\Delta(\theta_i)=\{z\mid|\arg z-\theta_i|<\frac{2\pi}{m},\quad|z|<1\}\ (i=0,1,2,\cdots,m-1),$$

then in it, at the least, exist a domain of circular sector $\Delta(\theta_{i_0})$ $(0<i_0<m-1)$, such that for any complex value $a$, there is

$$\overline{\lim_{r\to1}}\frac{\log^+n(r,\Delta(\theta_{i_0}),a)}{\log\frac{1}{1-r}}=\rho\qquad(0<\rho<+\infty).$$

at the most, it can exclude $2\nu$ numbers of Borel exceptional value $a$.

**Proof** Suppose the result is not true. Then exist $q=2\nu+1$ numbers of Borel exceptional value $\{a_i^j\}$ at least, $(j=1,2,\cdots,q)$, as for each domain of circular sector $\Delta(\theta_i)$ $(i=0,1,\cdots,m-1)$, let

$$\overline{\lim_{r\to1}}\frac{\log^+n(r,\ \Delta(\theta_i),a_i^j)}{\log\frac{1}{1-r}}<\rho\qquad(0<\rho<+\infty).$$

Then, it exists $r_0$ for any given $\varepsilon>0$. When $r>r_0$, as for $i$ and $j$, samely, we have

$$n(r,\ \Delta(\theta_i),a_i^j)<(\frac{1}{1-r})^{\rho-\varepsilon}\tag{5}$$

For any given positive integer $a$, let

$$\theta_{i,k}=\frac{2\pi i}{m}+\frac{2k\pi}{ma}$$

$0<i<m-1,\ 0<k<a-1;\ \theta_{i,0}=\theta_i,\ \theta_{i,m}=\theta_{i+1,0},$

$$\Omega_{i,k}=\{z\,|\,\theta_{i,k}\leqslant \arg z<\theta_{i,k+1},\ |z|<r^{\lambda}\}\quad (r>r_0).$$

Then

$$\{|z|<r^{\lambda}\}'=\sum_{k=0}^{a-1}\sum_{i=0}^{m-1}\Omega_{i,k}\ ,$$

so, it must have existed a $k_0$ $(0\leqslant k_0\leqslant a-1)$, where let $k_0=0$, such that

$$\sum_{i=0}^{m-1}n(\Omega_{i,0},\widetilde{R}_z)\leqslant \frac{1}{a}n(r^{\lambda},{}'R_z)$$

Then let domain of circular sector

$$\Delta_i^0=\{z\,|\,\theta_{i,0}\leqslant \arg z<\theta_{i+1,1},\ |z|<1\}$$

$$\overline{\Delta}_i=\{z\,|\frac{\theta_{i,0}+\theta_{i,1}}{2}\leqslant \arg z<\frac{\theta_{i+1,0}+\theta_{i+1,1}}{2},\ |z|<1\}$$

because $\Delta_i^0$ only cover $\Omega_{i,o}$ twice, then we have

$$\sum_{i=0}^{m-1}n(r^{\lambda},\ \Delta_i^0,\widetilde{R}_z)\leqslant (1+\frac{1}{a})n(r^{\lambda},\widetilde{R}_z) \tag{6}$$

Well, from lemma 2, as for each domain $\overline{\Delta}_i\subset\Delta_i^0$, we obtain

$$(q-2)S(r,\overline{\Delta}_i)\leqslant\sum_{j=1}^{q}n(r^{\lambda},\ \Delta_i^0,a_i^j)+n(r^{\lambda},\ \Delta_i^0,\widetilde{R}_z)+O(\log\frac{1}{1-r}).$$

Add up to $i$, and from (5) and (6), and

$$\sum_{i=0}^{m-1}S(r,\overline{\Delta}_i)=S(r)$$

we obtain

$$(q-2)S(r)\leqslant(1+\frac{1}{a})n(r^{\lambda},\widetilde{R}_z)+O((\frac{1}{1-r})^{\rho-\varepsilon})+O(\log\frac{1}{1-r}),$$

this time, because

$$\lim_{r\to1}\frac{(\frac{1}{1-r})^{\rho-\varepsilon}}{\log\frac{1}{1-r}}=(\rho-\varepsilon)\lim_{r\to1}(\frac{1}{1-r})^{\rho-\varepsilon}=\infty$$

then

$$(q-2)S(r)\leqslant(1+\frac{1}{a})n(r^{\lambda},\widetilde{R}_z)+O((\frac{1}{1-r})^{\rho-\varepsilon})$$

first of all, it is divided by $v_r$ on either side, then intergral from 0 to $r$, we obtain

$$(q-2)\frac{1}{v}\int_0^r\frac{S(r)}{r}dr\leqslant(1+\frac{1}{a})\frac{1}{v}\int_0^r\frac{n(t^{\lambda},\widetilde{R}_z)}{t}dt+O(\frac{1}{v}\int^r\frac{(\frac{1}{1-r})^{\rho-\varepsilon}}{r}dr).$$

From it, let $\tau=t^{\lambda}$, then $dt=\frac{1}{\lambda}\tau^{\frac{1}{\lambda}-1}d\tau$, we obtain

$$(q-2)T(r)\leqslant(1+\frac{1}{a})\frac{1}{\lambda}(\frac{1}{v}\int_0^{r^{\lambda}}\frac{n(\tau,\widetilde{R}_z)}{\tau}d\tau)+O(o(\frac{1}{1-r})^{\rho-\varepsilon}).$$

Then we obtain

$$(q-2)T(r)\leqslant\frac{1}{\lambda}(1+\frac{1}{a})N(r^{\lambda},R_z)+o((\frac{1}{1-r})^{\rho-\varepsilon}).$$

From (2), we obtain

$$(q-2)T(r)\leqslant 2(\nu-1)\left(1+\frac{1}{a}\right)\frac{1}{\lambda}T(r^{\lambda})+o\left(\left(\frac{1}{1-r}\right)^{\rho-\varepsilon}\right). \tag{7}$$

Now, let $\rho(\frac{1}{1-r})$ be an exact order of $W(z)$, and let $U(\frac{1}{1-r})=(\frac{1}{1-r})^{\rho(\frac{1}{1-r})}$ be a type function of $T(r)$. When $0<\rho<+\infty$, we have $\lim\limits_{r\to1}\rho(\frac{1}{1-r})=\rho$, and

$$\overline{\lim_{r\to1}}\frac{T(r)}{U(\frac{1}{1-r})}=1,\quad \overline{\lim_{r\to1}}\frac{U(\frac{1}{1-r^{\lambda}})}{U(\frac{1}{1-r})}=\frac{1}{\lambda^{\rho}},\quad \lim_{r\to1}\frac{(\frac{1}{1-r})^{\rho-\varepsilon}}{U(\frac{1}{1-r})}=0 \tag{8}$$

from (7), it is divided by $U(\frac{1}{1-r})$ on either side, after this, from it's superior limit, we obtain

$$(q-2)\overline{\lim_{r\to1}}\frac{T(r)}{U(\frac{1}{1-r})}\leqslant 2(\nu-1)\left(1+\frac{1}{a}\right)\frac{1}{\lambda}\overline{\lim_{r\to1}}\frac{T(r^{\lambda})}{U(\frac{1}{1-r})}$$

$$\leqslant 2(\nu-1)\left(1+\frac{1}{a}\right)\frac{1}{\lambda}\lim_{r\to1}\frac{T(r^{\lambda})}{U(\frac{1}{1-r^{\lambda}})}\overline{\lim_{r\to1}}\frac{U(\frac{1}{1-r^{\lambda}})}{U(\frac{1}{1-r})}$$

Then from (8), we can obtain $q-2\leqslant 2(\nu-1)(1+\frac{1}{a})\frac{1}{\lambda^{1+\rho}}$ It is in existence, for any number $(0<\lambda<1)$, and for any positive integer $a$. We can get $a$ sufficient big, and let $\lambda\to1$, then obtain $q\leqslant 2\nu$. But it is in contradiction with supposed $q=2\nu+1$.

Then lemma 3 is true.

## 3. Theorem and Proof

**Theorem** Let $W=W(z)$ be a $\nu$-value meromorphic algebroidal function in unit circle fixed by (1), it's positive finite order:

$$\overline{\lim_{r\to1}}\frac{\log^{+}T(r)}{\log\frac{1}{1-r}}=\rho\quad(0<\rho<+\infty),$$

then exist $\theta_0$ $(0\leqslant\theta_0<2\pi)$, such that for any given $\delta$ $(0<\delta<\frac{\pi}{2})$. in the domain of circular sector $\Delta=\{z|\ |\arg z-\theta_0|<\delta,\ |z|<1\}$, and for any complex value $a$, anytime there is

$$\overline{\lim_{r\to1}}\frac{\log^{+}n(r,\Delta,a)}{\log\frac{1}{1-r}}=\rho\quad(0<\rho<+\infty),$$

at most, it can be excluded $2\nu$ numbers of Borel exception value $a$. (About $\theta_0$ in the theorem, we can see, from definition, $e^{i\theta_0}$ is a Borel point of $W(z)$)

**Proof** From lemma 3, for any given positive integer $m$, anytime there exists a domain of circular sector $\Delta_m=\{z|\ |\arg z-\theta_m|<\frac{2\pi}{m},\ |z|<1\}$ such that for any complex value $a$, there is

$$\overline{\lim_{r\to 1}}\frac{\log^{+}n(r,\Delta_m,a)}{\log\frac{1}{1-r}}=\rho\quad(0<\rho<+\infty)\tag{9}$$

at most, it can be excluded $2\nu$ numbers of Borel exceptain value $a$. After choose the subsequence, we can suppose, when $m\to\infty$, $\theta_m\to\theta_0$. Then $e^{i\theta_0}$ is the Borel point.

In fact, for any given $\delta$ $(0<\delta<\frac{\pi}{2})$, in $\Delta$, if exist $2\nu+1$ numbers of Borel exception value $a$, let

$$\overline{\lim_{r\to 1}}\frac{\log^{+}n(r,\Delta,a)}{\log\frac{1}{1-r}}<\rho\quad(0<\rho<+\infty).$$

But we can get sufficiently big $m$, let $\Delta'_m\subset\Delta$, so, we have

$$\overline{\lim_{r\to 1}}\frac{\log^{+}n(r,\Delta_m,a)}{\log\frac{1}{1-r}}<\rho.$$

But it is in contradiction with (9).

Then finish the proof, and theorem is true.

### References


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## 单位圆内亚纯代数体函数的Borel点的存在性


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**摘要** 本文在单位圆内，对有穷正级的 $\nu$ 值亚纯代数体函数 $W(z)$ 定义了Borel 点，并证明了它的存在性。