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Approximation by Interpolating Polynomials in C_ω^p Spaces *

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Abstract In this paper we proved that for any function $f(x)\in C_\omega^p$, $1<p<+\infty$, $\omega(x)=1/\sqrt{1-x^2}$, there exists a series $\{\theta_n\}$ on $-\frac{\pi}{2}<\varphi_1<\theta<\varphi_2<\frac{\pi}{2}$, such that the order of approximation by interpolating polynomials of $f(x)$ at interpolation points $\{\theta_n\}$ can be estimated by the best approximation in C_ω^p spaces.

I. Introduction

We denote by C_ω^p the class of real functions satisfying the condition:

$$\int_{-1}^{1}|f(x)|^p\omega(x)\mathrm{d}x<+\infty,$$

where $0<p<+\infty$, $w(x)=1/\sqrt{1-x^2}$.

For $f(x)\in C_\omega^p$, the norm of which is defined by

$$\|f\|_p=\{\int_{-1}^{1}|f(x)|^p\frac{1}{\sqrt{1-x^2}}\mathrm{d}x\}^{1/p} \tag{1}$$

Let $_{n,k}(\theta)$, $1\leqslant k\leqslant n+1$ be the roots of the equation:

$$\cos[(n+1)\arccos x]-\sin\theta=0 \tag{2}$$

and $L_n(f;\theta_0;x)$ be the n-th Lagrange interpolationg polynomial to $f(x)$ at $\chi_{n,k}(\theta)$, $1\leqslant k\leqslant n+1$ (if $f[\chi_{n,k}(\theta_0)]=+\infty$ for some $\theta_0\in[\varphi_1,\varphi_2]$, we suppose $L_n(f;\theta_0,\chi_{n,k}(\theta_0))=+\infty$).

It is well known that for any fixed matrix of interpolation points there exists a continuous function, to which the interpolating polynomial doesn't converge uniformly. But in 1936 Erdös and Feldheim [1] proved the convergence in the mean of $L_n(f;0,x)$ in C_ω^p spaces. Leter, Askey ([2], [3]), Navai [4] got some results about the L^p convergence of interpolating polynomials under various weights. But all these result have been obtained under the assumption that the interpolated functions are continuous.

If we weaken the condition on $f(z)$ as only require that $f(z)$, $f(x)\in C_\omega^p$, there is no common matrix of interpolation points for all functions in C_ω^p, since functions in this class may not have definition at some points. However in this paper

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for each $f(x)\in C^p_\omega$ we still can construct a matrix of interpolation points throught the parameter θ and prove the convergence of the Lagrange interpolating polynomials of $f(x)$ at these interpolation points in this space rapidly.

Our main result is the following theorem.

Theorem 1 Suppose $f(x)\in C^p_\omega$, $1<p<+\infty$, $-\frac{\pi}{2}<\varphi_1<\theta<\varphi_2<\frac{\pi}{2}$, then there exists a sequence $\{\theta_n\}\in[\varphi_1,\varphi_2]$ such that

$$\|L_n(f;\theta_n;x)-f(x)\|_p<CE_{n,p}(f) \tag{3}$$

where C is a constant only depending on p,φ_1,φ_2, and

$$E_{n,p}(f)=\inf_{P_n\in\Pi_n}\|f(x)-P_n(x)\|_p$$

where the infimum is taken over all polynomials of degree at most n, which we denote by Π_n.

Here and after we denote by $C,C_1,C_2,\cdots$ the constant only depending on p, φ_1,φ_2, no matter how large they are, and we always assume $1<p<+\infty$, $-\frac{\pi}{2}<\varphi_1<\varphi_2<\frac{\pi}{2}$.

2. Some discussin on complex analysis

Let D be the unit disk $|z|<1$ in the complex plane.

Lemma 1 Suppose $\{z_j\}$, $1<j<N$, are distinct points in D, and

$$\min_k\prod_{\substack{j=1\\j\neq k}}^N\left|\frac{z_k-z_j}{1-\bar z_jz_k}\right|>\delta>0. \tag{4}$$

Then for any complex numbers $\{a_k\}$, $1<k<N$, there exists a function $g(z)$ analytic on $\overline{D}$ such that

1°. $g(z)=a_k$, $1<k<N$;

2°. $\int_{-\pi}^{\pi}|g(e^{it})|^p dt<C(p,\delta)\sum_{k=1}^N|a_k|^p(1-|z_k|)$,

where $C(p,\delta)$ is a constant only depending on p and δ.

The proof of Lemma 1 can be found in [5].

In 1981 Мартиросян[6] expresses the function $g(z)$ as follows:

$$\sum_{k=1}^N a_k((1-|z_k|^2)/(1-\bar z_kz))^2[F_k(z)B_k(z)]/[F_k(z_k)B_k(z_k)]$$

where

$$F_k(z)=\exp[-\sum_{j=k}^N(1-|z_j|^2)(1+\bar z_jz)/(1-\bar z_jz)]$$

$$B_k(z)=\prod_{\substack{j=1\\j\neq k}}^N(z-z_j)/(1-\bar z_jz).$$

By using Lemma 1 we can prove the following Lemma 2.

Lemma 2 Suppose $\theta\in[\varphi_1,\varphi_2]$ and $\xi_{n,k}(\theta)$, $k=\pm1,\cdots,\pm(n+1)$, are all the roots of the equation:

$$\mathrm{Re}\,\xi^{n+1}=\sin\theta$$

on $|\xi|=1$. Then for any polyomials $P_{2n+1}(z)\in\Pi_{2n+1}$, we have

$$\int_{-\pi}^{\pi}|P_{2n+1}(e^{it})|^p\mathrm{d}t<\frac{C_1}{n+1}\sum_{0<|k|<n+1}|P_{2n+1}(\xi_{n,k}(\theta))|^p. \tag{6}$$

The Marciekiewicz-Zygmund inequality [7] is a special case $\theta=0$ of this Lemma.

Proof Let $z_{n,k}(\theta)$, $k=\pm1,\cdots,\pm(n+1)$, be the roots of the equation:

$$\mathrm{Re}\ z^{n+1}-\rho_n\sin\theta=0 \tag{7}$$

on $|z|=\rho_n$ where

$$\rho_n=(1-\frac{1}{n+1})^{n+1} \tag{8}$$

We can get

$$z_{n,k}(\theta)=\rho_n^{1/(n+1)}\exp(\frac{4k\pi+\pi-2\theta}{2(n+1)}i) \tag{9}$$

$$z_{n,-k}(\theta)=\overline{z_{n,k}(\theta)}\ ,\ 1<k<n+1. \tag{10}$$

Thus we can see that $z_{n,k}(\theta)$ are also the roots of the following equation:

$$\omega_{n,\theta}(z)=z^{2(n+1)}-2\rho_n\sin\theta z^{n+1}+\rho_n^2 \tag{11}$$

For the sake of simplicity we use the notations $\omega_\theta(z)$, $z_k(\theta)$ and $\xi_k(\theta)$ to sub substitute $\omega_{n,\theta}(z)$, $z_{n,k}(\theta)$ and $\xi_{n,k}(\theta)$ respectively.

By calculating directly, we have

$$\sum_{0<|j|<(n+1)}\left|\frac{z-z_j(\theta)}{1-\overline{z_j}(\theta)z}\right|=\frac{|\omega_\theta(z)|}{|z|^{2n+2}}\prod_{0<|j|<(n+1)}\frac{1}{|z^{-1}-z_j(\theta)|}$$

$$=|\omega_\theta(z)|/(|z|^{2n+2}|\omega_\theta(z^{-1})|).$$

Let

$$A_k(\theta)=\prod_{\substack{0<|j|<(n+1)\\ j\neq k}}\left|\frac{z_k(\theta)-z_j(\theta)}{1-\overline{z_j(\theta)}z_k(\theta)}\right|. \tag{12}$$

Then

$$A_k(\theta)=\lim_{z\to z_k(\theta)}\left|\frac{1-\overline{z_k(\theta)}z}{z-z_k(\theta)}\right|\prod_{0<|z|<(n+1)}\left|\frac{z-z_j(\theta)}{1-\overline{z_j(\theta)}z}\right|$$

$$=\frac{1-|z_k(\theta)|^2}{|z_k(\theta)|^{2n+2}|\omega_\theta(z_k(\theta)^{-1})|}|\omega_\theta'(z_k(\theta))|.$$

Since

$$1-|z_k(\theta)|^2=1-(1-\frac{1}{n+1})^2 \tag{13}$$

$$|z_k(\theta)|^{2n+2}=\rho_n^2 \tag{14}$$

and in virtue of (11), we have

$$|\omega_\theta(z_k(\theta)^{-1})| \leqslant \rho_n^{-2} + 2|\sin\theta| + \rho_n^2 . \tag{15}$$

It is obvious that

$$|\omega'_\theta(z_k(\theta))| = 2(n+1)|z_k(\theta)|^n |(z_k(\theta))^{n+1} - \rho_n \sin\theta| . \tag{16}$$

Because of $\mathrm{Re}[z_k(\theta)]^{n+1} = \rho_n \sin\theta$ we have

$$|[z_k(\theta)]^{n+1} - \rho_n \sin\theta| = |I_m[z_k(\theta)]^{n+1}| = \rho_n \cos\theta .$$

Consequently fsom (13), (14), (15) and (16) we get

$$A_k(\theta) \geqslant \frac{2[1-(1-\frac{1}{n+1})^2](n+1)\cos\theta}{\rho_n^{1/(n+1)}(\rho_n^{-2}+2|\sin\theta|+\rho_n^2)} .$$

In virtue of $-\frac{\pi}{2} < \varphi_1 \leqslant \theta \leqslant \varphi_2 < \frac{\pi}{2}$ we get

$$A_k(\theta) \geqslant C > 0 . \tag{17}$$

For $P_{2n+1}(\xi) \in \Pi_{2n+1}$, let

$$P_{2n+1}(\xi_k(\theta)) = a_k(\theta) \tag{18}$$

and

$$Q_{2n+1}(z) = P_{2n+1}(\rho_n^{-[1/(n+1)]} z) .$$

Then

$$\{\int_{-\pi}^{\pi} |P_{2n+1}(e^{it})|^p \mathrm{d}t\}^{1/p} \leqslant \rho_n \{\int_{-\pi}^{\pi} |Q_{2n+1}(e^{it})|^p \mathrm{d}t\}^{1/p}$$

$$\leqslant 2e\{\int_{-\pi}^{\pi} |Q_{2n+1}(e^{it})|^p \mathrm{d}t\}^{1/p} . \tag{19}$$

Besides we have

$$Q_{2n+1}(z_k(\theta)) = a_k(\theta), \quad 1 \leqslant |k| \leqslant n+1 . \tag{20}$$

In virtue of Lemma 1 (see (17)) there exists a function $g_\theta(z)$ analytic on $\overline{D}$ such that

$$g_\theta(z_k(\theta)) = a_k(\theta), \quad 1 \leqslant |k| \leqslant n+1 \tag{21}$$

and

$$\int_{-\pi}^{\pi} |g_\theta(e^{it})|^p \mathrm{d}t \leqslant C_3 \sum_{0<|k|\leqslant(n+1)} |a_k(\theta)|^p (1-|z_k|) \leqslant \frac{C_4}{n+1} \sum_{0<|k|\leqslant(n+1)} |a_k(\theta)|^p . \tag{22}$$

It is clear that $Q_{2n+1}(z)$ is the interpolating polynomial of $g_\theta(z)$ at $\{z_k(\theta)\}$, $1 \leqslant |k| \leqslant n+1$. Then we have

$$g_\theta(z) - Q_{2n+1}(z) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{\omega_\theta(z)}{\omega_\theta(\xi)} \frac{g_\theta(\xi)}{\xi - z} \mathrm{d}\xi, \quad |z| < 1 . \tag{23}$$

By the (p, p) property of Cauchy integral operator we have

$$\{\int_{-\pi}^{\pi} |g_\theta(e^{it}) - Q_{2n+1}(e^{it})|^p \mathrm{d}t\}^{1/p}$$

$$\leqslant C_5 \sup_{|z|=1} |\omega_\theta(z)| \{\int_{-\pi}^{\pi} \left|\frac{g_\theta(e^{it})}{\omega_\theta(e^{it})}\right|^p \mathrm{d}t\}^{1/p}$$

$$\leqslant C_5 \sup_{|z|=1} |\omega_\theta(z)| / \inf_{|\xi|=1} |\omega_\theta(\xi)| \{\int_{-\pi}^{\pi} |g_\theta(e^{it})|^p \mathrm{d}t\}^{1/p} . \tag{24}$$

In virtue of (11) we have

$$|\omega_\theta(z)|\leqslant|z|^{2n+2}+2\rho_n|\sin\theta||z|^{n+1}+\rho_n^2$$
$$\leqslant 1+\rho_n+\rho_n^2,\quad |z|=1,\quad -\frac{\pi}{2}<\varphi_1\leqslant\theta\leqslant\varphi_2<\frac{\pi}{2}$$

and

$$|\omega_\theta(\xi)|^{-1}\leqslant(|\xi|^{2n+2}-2\rho_n|\sin\theta||\xi|^{n+1}-\rho_n^2)^{-1}$$
$$\leqslant\frac{1}{1-2\rho_n-\rho_n^2},\quad |\xi|=1,\quad -\frac{\pi}{2}<\varphi_1\leqslant\theta\leqslant\varphi_2<\frac{\pi}{2}.$$

Since when n tends to $+\infty$, we have $\rho_n\to e^{-1}$, then for sufficient lagre n

$$\frac{1}{1-2\rho_n-\rho_n^2}\geqslant\frac{1}{2(1-2e^{-1}-e^{-2})}>0.$$

Hence

$$\sup_{|z|=1}|\omega_\theta(z)|/\inf_{|\xi|=1}|\omega_\theta(\xi)|\leqslant C_6.$$

Consequently in virtue of (24) we arrive

$$\{\int_{-\pi}^{\pi}|g_\theta(e^{it})-Q_{2n+1}(e^{it})|^p\mathrm{d}t\}^{1/p}\leqslant C_5C_6\{\int_{-\pi}^{\pi}|g_\theta(e^{it})|^p\mathrm{d}t\}^{1/p}.$$

Thus by (22) we get

$$\{\int_{-\pi}^{\pi}|Q_{2n+1}(e^{it})|^p\mathrm{d}t\}^{1/p}\leqslant C_7\{\int_{-\pi}^{\pi}|g_\theta(e^{it})|^p\mathrm{d}t\}^{1/p}$$
$$\leqslant C_8\{\frac{1}{n+1}\sum_{0<|k|\leqslant(n+1)}|a_k(\theta)|^p\}^{1/p}. \tag{25}$$

By (19), (25) and (18) at last we get

$$\{\int_{-\pi}^{\pi}|P_{2n+1}(e^{it})|^p\mathrm{d}t\}^{1/p}\leqslant 2e\{\int_{-\pi}^{\pi}|Q_{2n+1}(e^{it})|^p\mathrm{d}t\}^{1/p}$$
$$\leqslant C_9\{\frac{1}{n+1}\sum_{0<|k|\leqslant(n+1)}|a_k(\theta)|^p\}^{1/p}=C_9\{\frac{1}{n+1}\sum_{0<|k|\leqslant(n+1)}|P_{2n+1}(\xi_{n,k}(\theta))|^p\}^{1/p}$$

Thus we complete the proof of Lemma 2.

3. Proof of Theorem 1

By (2) we can get

$$\chi_{n,k}(\theta)=\cos\frac{4k\pi+\pi-2\theta}{2(n+1)},\quad 1\leqslant k\leqslant n+1. \tag{26}$$

Also we have

$$\mathrm{Re}\,\xi_{n,k}(\theta)=\chi_{n,k}(\theta),\quad k=\pm1,\pm2,\cdots,\pm(n+1). \tag{27}$$

For $f(x)\in C_\omega^p$. we can verify

$$\|f\|_p=\{\frac{1}{2}\int_{-\pi}^{\pi}|f(\cos t)|^p\mathrm{d}t\}^{1/p}. \tag{28}$$

Since

$$L_n(f;\theta;\cos t)=L_n(f;\theta;\frac{1}{2}(z+z^{-1})),\quad z=e^{it} \tag{29}$$

are polynomials with respect to $z+z^{-1}$, the degree of which doesn't exceed n, we have

$$H_n(z):=z^n L_n(f;\theta;\frac{1}{2}(z+z^{-1}))\in\Pi_{2n}$$

and

$$\begin{aligned}H_n(\xi_{n,k}(\theta))&=[\xi_{n,k}(\theta)]^n L_n(f;\theta;\operatorname{Re}\xi_{n,k}(\theta))\\&=[\xi_{n,k}(\theta)]^n L_n(f;\theta;\chi_{n,|k|}(\theta))\\&=[\xi_{n,k}(\theta)]^n f(\chi_{n,|k|}(\theta)),\quad k=\pm1,\pm2,\cdots,\pm(n+1).\end{aligned}$$

By Lemma 2 we have

$$\int_{-\pi}^{\pi}|H_n(e^{it})|^p\mathrm{d}t\le\frac{C}{n+1}\sum_{0<|j|<(n+1)}|f(\chi_{n,|j|}(\theta))|^p=\frac{2C}{n+1}\sum_{k=1}^{n+1}|f(\chi_{n,k}(\theta)|^p.$$

Since

$$\int_{-\pi}^{\pi}|L_n(f;\theta;\cos t)|^p\mathrm{d}t=\int_{-\pi}^{\pi}|H_n(e^{it})|^p\mathrm{d}t,$$

then

$$\int_{-\pi}^{\pi}|L_n(f;\theta;\cos t)|^p\mathrm{d}t\le\frac{2C}{n+1}\sum_{k=1}^{n+1}|f(\chi_{n,k}(\theta)|^p. \qquad (30)$$

Let $P_n(x)\in\Pi_n$ be the best approximation polynomial, it means

$$\|f(x)-P_n(x)\|_p=E_{n,p}(f).$$

Since

$$L_n(P_n;\theta;x)=P_n(x),$$

we have

$$\begin{aligned}&\|f(x)-L_n(f;\theta;x)\|_p\\&=\|f(x)-P_n(x)+L_n(P_n-f;\theta;x)\|_p\le E_{n,p}(f)+\|L_n(f-P_n;\theta;x)\|_p\\&\le E_{n,p}(f)+\{\frac{2C}{n+1}\sum_{k=1}^{n}|f(\chi_{n,k}(\theta))-P_n(\chi_{n,k}(\theta))|^p\}^{1/p}.\end{aligned}$$

Then

$$\begin{aligned}&\{\int_{\varphi_1}^{\varphi_2}\|f(x)-L_n(f;\theta;x)\|_p^p\mathrm{d}t\}^{1/p}\\&\le\pi E_{n,p}(f)+\{\frac{2C}{n+1}\sum_{k=1}^{n+1}\int_{\varphi_1}^{\varphi_2}|f(\chi_{n,k}(\theta))-P_n(\chi_{n,k}(\theta))|^p\mathrm{d}\theta\}^{1/p}\\&\le\pi E_{n,p}(f)+\{\frac{2C}{n+1}\sum_{k=1}^{n+1}\int_{\varphi_1}^{\varphi_2}|f(\cos\frac{4k\pi+\pi-2\theta}{2(n+1)})-P_n(\cos\frac{4k\pi+\pi-2\theta}{2(n+1)})|^p\mathrm{d}\theta\}^{1/p}\\&\le\pi E_{n,p}(f)+\{C_{10}\int_{-\pi}^{\pi}|f(\cos t)-P_n(\cos t)|^p\mathrm{d}t\}^{1/p}\le C_{11}E_{n,p}(f).\end{aligned}$$

Let

$$G_n(f)=\{\theta|\varphi_1\le\theta\le\varphi_2,\ \|f(x)-L_n(f;\theta;x)\|_p\le\frac{3}{\varphi_2-\varphi_1}C_{11}E_{n,p}(f)\},$$

and denote by $|G_n(f)|$ the Lebesque measure of $G_n(f)$, then

$$\begin{aligned}&(\frac{3}{\varphi_2-\varphi_1})^{1/p}C_{11}E_{n,p}(f)[\varphi_2-\varphi_1-|G_n(f)|]^{1/p}\\&\le\{\int_{[\varphi_1,\varphi_2]\setminus G_n(f)}\|f(x)-L_n(f;\theta;x)\|_p^p\mathrm{d}\theta\}^{1/p}\le C_{11}E_{n,p}(f).\end{aligned}$$

Thus we can get

$$|G_n(f)|>(\varphi_2-\varphi_1)-\frac{\varphi_2-\varphi_1}{3}=\frac{2(\varphi_2-\varphi_1)}{3}. \tag{31}$$

That means $G_n(f)$ is not empty, so there exists $\{\theta_n\}\in G_n(f)$ such that

$$\|f(x)-L_n(f;\theta;x)\|_p<\frac{3}{\varphi_2-\varphi_1}C_{11}E_{n,p}(f). \tag{32}$$

It is namely (3), thus we complete the proof of Theorem 1.

Theorem 2 Suppose $f_1(x), f_2(x)\in C_\omega^p$, $1<p<+\infty$, $-\frac{\pi}{2}<\varphi_1<\varphi_2<\frac{\pi}{2}$, then there exists a sequence $\{\theta_n\}\in[\varphi_1,\varphi_2]$ such that (3) is valid for both $f_1(z)$ and $f_2(z)$.

In fact because of (31) we have

$$|G_n(f_1)\cap G_n(f_2)|>\frac{1}{3}(\varphi_2-\varphi_1).$$

Thus there exists $\theta_n\in\{G_n(f_1)\cap G_n(f_2)\}$ such that (32) is valid for both $f_1(z)$ and $f_2(z)$.

It is obvious that we can extend Theorem 2 to the case of the finite number of functions $f_1, f_2, \cdots, f_m\in C_\omega^p$.

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C_ω^p 空间中的插值多项式逼近

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摘要 本文证明了对任意函数 $f(z)\in C_\omega^p$，其中 $1<p<+\infty$，$\omega(x)=\frac{1}{\sqrt{1-x^2}}$，在 $-\frac{\pi}{2}<\varphi_1<\theta<\varphi_2<\frac{\pi}{2}$ 上存在序列 $\{\theta_n\}$，使得 $f(z)$ 在 $\{\theta_n\}$ 上的插值多项式在 C_ω^p 中的逼近阶能用其最佳逼近来估计.