Journal of Mathematical Research and Exposition
Vol.10 No.1 Feb. 1990

Critical Points at Infinity and Limit Cycle of Lienard Equation*

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We consider the Lienard equation

$$\ddot{x}+f(x)\dot{x}+g(x)=0. \tag{1}$$

or its equivalent representation

$$\dot{x}=y,\ \dot{y}=-f(x)y-g(x) \tag{2}$$

where $f(x)=\sum_{s=0}^{m}a_sx^s$, $g(x)=\sum_{s=0}^{n}b_sx^s$ are polynomials and $m\geqslant n>0$.

R. Gomory[1] had completely analyzed the nature of the critical points at infinity of the system (2), and deduced about the behavior in the large of trajectories by the leading coefficients a_m and b_n of the polynomials as well as by the parity of m and n. We will in the paper discuss the existence of limit cycles for the system (2) on the basis of paper [1]. The new criterions are given for the existence of limit cycles of the system (2). Some earlier results (see, [3, chap.4 and 5]) can or not be used in order to decide the existence of limit cycles, and even if they can be used as the criterions, however, the examining steps are fairly complex. But our results can easily be applied. In addition, we give a new criterion using theory of Hopf bifurcation for the existence of limit cycle of some complex system (2). In the final part of this paper, some examples are given to illustrate cases in which the new criterions can easily be applied.

§ 1 The existence of limit cycles

We can easily prove the following results on the basis of paper [1].

Theorem 1 Let the system (2) exist the singular points in the finite plane. Then we have either

1° the index sum of the singular points is 1 (or −1) as n odd and $b_n>0$ (or <0), or

2° the index sum of the singular points is zero as n even.

* Received Mar. 14, 1988.

In the following, we discuss the existence of limit cycle for the system (2). We knew that a limit cycle must contain singularities with an index sum of 1. Thus we may obtain the following result by Theorem 1.

Theorem 2 If n odd $b_n<0$, or if n even, then the system (2) has no limit cycle that it contain all singularities in the finite plane.

Especially when the system (2) only has a single singularity, it can't exist limit cycle under the conditions of Theorem 2.

It will be convenient to use the following system which is equivalent to (2)

$$\dot{x}=y,\ \dot{y}=-b_0-b_1x-a_0y+Q_2(x,y) \tag{3}$$

where $Q_2(x,y)$ is a polynomial of degree $\geqslant 2$.

By paper [3], we have the following possibilities to (3).

Case 1 If $b_0=0$, $b_1\neq 0$ and $a_0\neq 0$, then the singular point $O(0,0)$ is a node or focus point for $b_1>0$, and it is stable for $a_0>0$ and unstable for $a_0<0$ as well as $O(0,0)$ is a saddle point for $b_1<0$.

Case 2 If $b_0\neq 0$, then let the singular point of the system (3) be $P_1(r,0)$ $(r\neq 0)$. For $f(r)\neq 0$ and $g'(r)\neq 0$, we replace x by $\xi+r$ and y by η to obtain the system

$$\dot{\xi}=\eta,\ \dot{\eta}=-g'(r)\xi-f(r)\eta+\overline{Q}_2(\xi,\eta). \tag{4}$$

where $\overline{Q}_2(\xi,\eta)$ is a polynomial of degree $\geqslant 2$.

For the system (4) we easily know that the singular point $P_1(r,0)$ is a node or focus point as $g'(r)>0$, and it is stable for $f(r)>0$ and unstable for $f(r)<0$ as well as it is a saddle point for $g'(r)<0$.

As before, if the system (2) only has a single singularity in the finite plane, then we can conclude that the system (2) has no limit cycle near the singular point as $b_1<0$ (or $g'(r)<0$), no matter whether m and n are odd or even. If $b_1>0$ (or $g'(r)>0$), we obtain a criterion for the existence of limit cycle of the system (2) by the results of paper [1].

Theorem 3 Let $O(0,0)$ (or $P_1(r,0)$) is a single singularity, and $xg(x)>0$ if $x\neq 0$. Suppose n odd, m even, $b_n>0$, $b_1>0$ (or $g'(r)>0$). Then either

1° the system (2) has at least a stable limit cycle as $a_m>0$ and $a_0<0$ (or $f(r)<0$), or

2° the system (2) has at least an unstable limit cycle as $a_m<0$ and $a_0>0$ (or $f(r)>0$).

Proof We first prove 1°. By case 1 and case 2, we know that the singular point $O(0,0)$ (or $P_1(r,0)$) of the system (2) is an unstable node or focus poins under the conditions of theorem and it is a repellor. In order to analyze the nature of the critical points at infinity of (2), we adopt homogeneous coordinates (x,y,z), i.e., respectively replacing x and y by x/z and y/z. With $dt=z^m d\tau$

for the region covered by $(x, 1, z)$ we obtain

$$\begin{cases} \dfrac{dz}{d\tau} = z(f^H(x, z) + z^{m-n+1} g^H(x, z)) \\ \dfrac{dx}{d\tau} = x(f^H(x, z) + z^{m-n+1} g^H(x, z)) + z^m \end{cases} \tag{5}$$

and for the region with coordinates $(1, y, z)$, using $dt = z^m d\sigma$

$$\begin{cases} \dfrac{z}{\sigma} = -z^{m+1} y \\ \dfrac{y}{\sigma} = -f^H(1, z) y - z^{m-n-1} g^H(1, z) - z^m y^2 \end{cases} \tag{6}$$

where $f^H(x, z)$, $g^H(x, z))$ denote f and g made homogeneous with z. It is evident from (5) and (6) that $z = 0$ is a trajectory connecting critical points, and that the only critical points on $z = 0$ are $P(0,1,0)$ and $Q(1,0,0)$. By ths results of paper [1], we know that $P(0,1,0)$ is a (*non*-simple) unstable node and $Q(1,0,0)$ is a saddle under the supposing of theorem. Because $z = 0$ is a trajectory, all the trajectory at P(except for the equater) tend to the finite plane from the equator. Two boundaries at Q leave from Q, and one of them enter the upper half poincare sphere. Because m is even, the new critical points $\bar{P}(0, -1, 0)$ and $\bar{Q}(-1, 0, 0)$, together with its diametrical opposite P and Q appear on the equator, are respectively possessed of the same stable nature with P and Q. Therefore the singular points at infinity are all repellors. The equator can play a role of the outer boundary. So the Poincare-Bendixson theorem now implies that system (2) has at least one stable limit cycle (see Fig.1).

2° As opposed to case 1°, the singular point $O(0,0)$ (or $P_1(r, 0)$) of (2) is a stable node or focus point and it is an attractor. Moreover, negative half-trajectories which start at P(or $\bar{P}$) and Q(or $\bar{Q}$) at infinity enter the finite plane and positive half-trajectories tend to the critical points at infinity. Hence, the system (2) has at least one unstable limit cycle, and the proof is complete (see Fig.2).

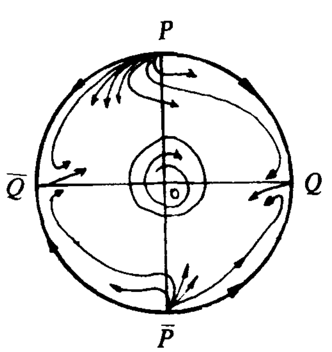

Fig.1

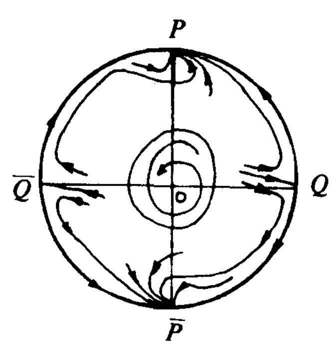

Fig.2

In addition, if m and n odd, $b_n>0, a_m>0$ or $a_m<0$, then the critical point P at infinity has both elliptic and hyperbolic sectors and Q a saddle point. By Theorem 1, we know that the index sum of system (2) is 1 in the finite plane. Though two cases are more complex, we obtain the following conclusion using the theory of Hopf bifurcation,

Theorem 4 Let $O(0,0)$ be a singular point of (2) and $xg(x)>0$ if $x\neq 0$; m, n odd; $b_n>0$; $b_1>0$; $a_m>0$ or $a_m<0$. Then, for $|a_0|>0$ small enough, near the origin the system (2) has at least one limit cycle which it is stable as $a_0<0$ and $b_1a_2<a_1b_2$ or unstable as $a_0>0$ and $b_1a_2>a_1b_2$.

Proof We may rewrite the system (2) in the form of an equivalent system

$$\dot{x}=\sqrt{b_1}\,y,\quad \dot{y}=-\sqrt{b_1}\,x-a_0y-\sum_{s=1}^{m}a_sx^sy-\frac{1}{\sqrt{b_1}}\sum_{s=2}^{n}b_sx^s \tag{7}$$

For convenience, we may replace $\sqrt{b_1}\,dt$ by $d\tau$ and still denote $d\tau$ with dt. Let $\lambda=-\dfrac{a_0}{\sqrt{b_1}}$, $a_i'=\dfrac{a_i}{\sqrt{b_1}}$ $(i=1,\cdots,m)$, $b_j'=\dfrac{b_j}{b_1}$ $(j=2,\cdots,n)$. Therefore by the system (7), we obtain the equivalent system

$$\dot{x}=y,\quad \dot{y}=-x+\lambda y-\sum_{s=1}^{m}a_s'x^sy-\sum_{s=2}^{n}b_s'x^s \tag{8}$$

where $b_n'>0$. Without loss of generality, we may assume $a_m'>0$. If $a_m'<0$, then we exchange x by $-x'$, y by $-y'$ and note m, n odd as well as $b_n'>0$ invariant. Hence, we may change the case $a_m'<0$ into $a_m''=-a_m'>0$. Let λ be a parameter, and $O(0,0)$ be the singular point of system (8). The matrix of first partial derivatives of (8) at $O(0,0)$ is

$$\begin{pmatrix} 0 & 1\\ -1 & \lambda\end{pmatrix}$$

whose eigenvalues are given by

$$\frac{1}{2}(\lambda\pm(\lambda^2-4)^{\frac{1}{2}}=\frac{1}{2}\lambda\pm i(1-\frac{\lambda^2}{4})^{\frac{1}{2}},\quad \text{for}\quad |\lambda|<2.$$

we conclude that this singularity satisfies

$O(0,0)$ is an unstable focus, if $\lambda>0$, i.e. $a_0<0$.

For $\lambda=0$, i.e. $a_0=0$, we rewrite the system (8) to obtain

$$\dot{x}=y,\quad \dot{y}=-x-\sum_{s=1}^{m}a_s'x^sy-\sum_{s=2}^{n}b_s'x^s \tag{9}$$

Consider the following formal series

$$F(x,y)=x^2+y^2+F_3(x,y)+F_4(x,y)+\cdots$$

where $F_k(x,y)$ $(k=3,4,\cdots)$ are made homogeneous of x and y.

We now calculate

$$\left.\frac{dF}{dt}\right|_{(9)}=(2x+\sum_{s=3}^{\infty}\frac{\partial F_s}{\partial x})y+(2y+\sum_{s=3}^{\infty}\frac{\partial F_s}{\partial y})(-x-\sum_{s=1}^{m}a_s'x^sy-\sum_{s=2}^{n}b_s'x^s) \tag{10}$$

Let the cubic term be zero in the right hand of above equationn

$$y\frac{\partial F_3}{\partial x}-x\frac{\partial F_3}{\partial y}-2a_1'xy^2-2b_2'x^2y=0.$$

By introducing extreme coordinate $x=\rho\cos\theta$, $y=\rho\sin\theta$ and eliminating ρ^3, the the above equation becomes

$$\frac{dF_3(\cos\theta,\sin\theta)}{d\theta}=-2a_1'\cos\theta\sin^2\theta-2b_2'\cos^2\theta\sin\theta$$

Because $\int_0^{2\pi}(-2a_1'\cos\theta\sin^2\theta-2b_2'\cos^2\theta\sin\theta)d\theta=0$, we obtain

$$F_3(\cos\theta,\sin\theta)=\frac{2}{3}(b_2'\cos^3\theta-a_1'\sin^3\theta)$$

and the third homogeneous function

$$F_3(x,y)=\frac{2}{3}(b_2'x^3-a_1'y^3)$$

Let the quartic term be zero in the right hand of (10), i.e.

$$y\frac{\partial F_4}{\partial x}-x\frac{\partial F_4}{\partial y}-2a_2'x^2y^2-2b_3'x^3y-a_1'xy\frac{\partial F_3}{\partial y}-b_2'x^2\frac{\partial F_3}{\partial y}=0$$

Using the same method as before, we have

$$\frac{dF_4(\cos\theta,\sin\theta)}{d\theta}=2a_2'\cos^2\theta\sin^2\theta+2b_3'\cos^3\theta\sin\theta-2a_1'^2\cos\theta\sin^3\theta$$
$$-2a_1'b_2'\cos^2\theta\sin^2\theta$$

Let the right hand of this equation be $H_4(\cos\theta,\sin\theta)$. Because

$$\int_0^{2\pi}H_4(\cos\theta,\sin\theta)d\theta=\frac{\pi}{2}(a_2'-a_1'b_2')\neq 0,$$

F_4 satisfies

$$\frac{dF_4(\cos\theta,\sin\theta)}{d\theta}=H_4(\cos\theta,\sin\theta)+C_4$$

where $C_4=\frac{1}{4}(a_2'-a_1'b_2')$.

Let $\Phi(x,y)=x^2+y^2+F_3(x,y)+F_4(x,y)$. Then we get

$$\left.\frac{d\Phi}{dt}\right|_{(9)}=\frac{1}{4}(a_2'-a_1'b_2')\rho^4+o(\rho^4)$$

So for $a_2'<a_1'b_2'$, i.e., $b_1a_2<a_1b_2$, $O(0,0)$ is a stable focus point of the central form. Hence, by the theorm of Hopf bifurcation (see, [4, chap. 7]), when $b_1a_2<a_1b_2$, we can conclude that the system (2) has at least one stable limit cycle near the origin for $\lambda>0$ (i.e., $a_0<0$) small enough (see Fig 3).

For the case $a_0>0$, we can change variables $x=-x'$, $t=-t'$ and reduce to the case of $a_0'=-a_0<0$. Because of m and n odd, a_m and b_n don't change sign. But $b_1a_2<a_1b_2$ become $b_1a_2>a_1b_2$. In fact, for $a_0>0$, $O(0,0)$ is a stable focus point of the system (7), and for $a_0=0$, $b_1a_2>a_1b_2$, $O(0,0)$ is an unstable focus of the central form of the system (9). Therefore by the theory of Hopf bifurcation, when $a_0>0$ small enough and $b_1a_2>a_1b_2$, we can conclude that system (2) has at least an unstable limit cycle near the origin (see, Fig .4).

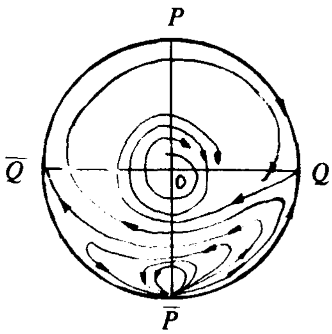

Fig.3 $a_m>0, a_0<0$.

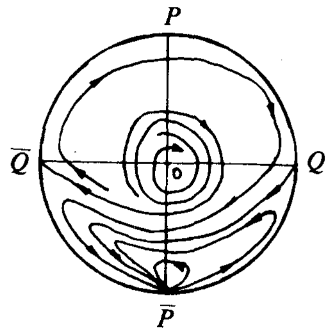

Fig.4 $a_m>0, a_0>0$.

For the case $a_m<0$, respectively we may get the phase portraits through inverting the direction of trajectory in Fig.3 and Fig.4 as well as turning $180°$ for the whole figure with respect to the x-axis,

We can obtain an analogous result for system (4).

In the following, we give some examples to illustrate cases in which our results can easily be applied .

§2 . Examples

Example 1 Let the equation

$$\ddot{x}+(x^4+4x+3)\dot{x}-x^3+2x^2-3x=0 \tag{11}$$

Cleaely $x=0$ is a simple root of the equation $-x^3+2x^2-3x=0$. Because $n=3$ odd and $b_3=-1<0$, by Theorem 2, we can conclude that the equation (11) has not limit cycle .

Example 2 Let the equation

$$\ddot{x}+(1-(m+1)x^m)\dot{x}+2x=0. \tag{12}$$

where $m\geqslant 2$ even .

Because $f(x)=1-(m+1)x^m$, $g(x)=2x$, m even, $n=1$ odd, $a_m=-(m+1)<0$, $a_0=1>0$, $b_n=b_1=2>0$, by Theorem 3, we know that the equation (12) has at least an unstable limit cycle.

We may verify that Драгилёв as well as Филиппов, Levinson and Smith's criterias [3] fail for the existence of limit cycle of the equation (12). But we can use Theorem 1 of the paper [5] as a criterion. However, the examining steps is fairly complex. Because of limit of the paper, we don't write out the concrete steps .

Acknowledgmnts *I wish to expres s my gretitude to professor Yu Shuxiang for the encouragement and help.*

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Lienard方程的无穷远奇点和极限环

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摘要 本文利用文[1]中关于Lienard方程

$$\ddot{x}+f(x)\dot{x}+g(x)=0.$$

的极限环的存在性，这里$f(x)$，$g(x)$为多项式，给出了直接利用多项式的系数就可以判断某些Lienard方程存在或不存在极限环的条件。

在无穷远奇点的特性进一步研究它的极限环的存在性，这里$f(x)$，$g(x)$为多项式，给出了直接利用多项式的系数就可以判断某些Lienard方程存在或不存在极限环的条件。