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A Note on a Paper of Dickmeis etc*

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Let X be a Banach space, X^* be the class of all bounded and sublinear functionals T on X, i.e.

(1) $|T(f+g)|\le|Tf|+|Tg|$ for all $f,g\in X$;

(2) $|T(af)|=|a||Tf|$ for all numbers a and $f\in X$;

(3) $\|T\|_{X^*}=\sup\limits_{\substack{f\in X\\ \|f\|_X\le 1}}|Tf|<\infty$.

In [1], W.Dickmeis, R.J.Nessel and E. van Wickeren proved the following theorem.

Theorem 1 Let $\{\psi_n\}$ and $\{\rho_n\}$ be the positive decreasing nullsequences. If $T_n,R_n,S_n\in X^*$, $h_n\in X$ with

(4) $\|h_n\|_X\le C_1$, $\|T\|_{X^*}\le C_2$ for $n=1,2,\cdots$

possess the following properties:

(5) $|T_nh_j|\le C_3\psi_n\psi_j^{-1}$ for $n,j=1,2,\cdots$

(6) $|R_nh_j|\le C_4\rho_n\psi_j^{-1}$ for $j\le n$ and $n=1,2,\cdots$,

(7) $\underline{\lim}_{n\to\infty}|S_nh_n|\ge C_5>0$,

(8) $|S_nh_j|\le A_j\psi_n$.

Then for any modulus of continuity $\omega(t)$ with

(9) $\lim\limits_{t\to 0+}\omega(t)t^{-1}=\infty$

there exists an element $f_\omega\in X$ such that

(10) $|T_nf_\omega|\le C_6\omega(\psi_n)$.

(11) $\overline{\lim}_{n\to\infty}|S_nf_\omega|(\omega(\psi_n))^{-1}\ge C_5$,

and

(12) $\overline{\lim}_{n\to\infty}|S_nf_\omega||R_nf_\omega|^{-1}\rho_n\psi_n^{-1}\ge C_7>0$.

We find, that this theorem is not such accurate, some conditions are not necessary. Below we shall improve Theorem 1, and give it a constructive proof. In our result, we cancel the condition (8) and the monotone condition of the

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sequence $\{\psi_n\}$ and $\{\rho_n\}$, and use a weaker condition instead of (1) for S_n.

At beginning we give the following definitions.

Denote by X^{**} the class of functionals T on X satisfying the conditions (2), (3) and

$$|Tf-Tg|\le M_T\|f-g\|_X .$$

for all $f, g\in X$, where M_T is a constant only depending upon T.

Let $V_n\in X^{**}$ and for some element f_k from a set $F\subseteq X$ and a null sequence $\{\eta_n\}$ it is valid that $\overline{\lim\limits_{n\to\infty}}|V_nf_k|/|\eta_n|=0$. If for any sequence $\{h_n\}\subseteq X$ with the property $|V_nh_n|\ge C>0$ one has

$$|V_n(h_n+\sum_{f_k\in F}{}'\frac{a_k}{|\eta_n|}f_k)|\ge C-o(1) \text{ as } n\to\infty,$$

where $\sum\limits_{f_k\in F}{}'a_kf_k$ is some finite combination of f_k, then we say that $\{V_n\}$ possesses the property (Z) to F.

Indeed, if $\{V_n\}\subseteq X^*$, then $\{V_n\}\subseteq X^{**}$possessing the property (Z) to X.

Theorem 2 Let $\{\psi_n\}$ and $\{\rho_n\}$ be the positive null-sequances. If for T_n, $R_n\in X^*$, $S_n\in X^{**}$ possessing the property (Z), to $\{h_k\}, h_n\in X$ with (4) the conditons (5), (6) and

(13) $\overline{\lim\limits_{n\to\infty}}|S_nh_n|\ge C>0$

are valid, then for any modulus of continuity $\omega(t)$ with (9) there is an element $f_\omega\in X$ satisfying (10—12).

Proof At first we indicate that, if for some fixed k_0 there is h_{k_0} satisfying $\overline{\lim\limits_{n\to\infty}}|S_nh_{k_0}|/\omega(\psi_n)>0$, then from the conditions (5, 6, 9, 13) one can deduce that

$$|T_nh_{k_0}|\le C_3\psi_{k_0}^{-1}\psi_n\le C_8\,\omega(\psi_n),\ n=1,2,\cdots,$$

$$\overline{\lim_{n\to\infty}}|S_nh_{k_0}|\ |R_nh_{k_0}|^{-1}\rho_n\psi_n^{-1}=\overline{\lim_{n\to\infty}}\frac{S_nh_{k_0}}{\omega(\psi_n)}\left(\frac{R_nh_{k_0}}{\rho_n\psi_{k_0}^{-1}}\right)^{-1}\frac{\omega(\psi_n)}{\psi_n}\ \frac{1}{\psi_{k_0}}=\infty,$$

so taking $f_\omega=\max\{1,\overline{\lim\limits_{n\to\infty}}|S_nh_{k_0}|/\omega(\psi_n)^{-1}\}C_5h_{k_0}$, we complete the proof of Theorem 2.

Now suppose that for any fixed k there is

$$\overline{\lim_{n\to\infty}}|S_nh_k|/\omega(\psi_n)=0.$$

Without lossing the generality we can assume that $|S_nh_n|\ge C_5$, otherwise we can turn to consider some subclass $\{h_{n_j}\}$ of $\{h_n\}$. Select $\{n_j\}$ to construct the element f_ω as follows. Set $n_1=1$, suppose that $n_1, n_2, \cdots, n_k$ are given. Choose n_{k+1} satisfying the following properties (due to conditions of Theorem 2):

(14) $\psi_{n_{k+1}}\le\psi_{n_k}$,

(15) $\omega(\psi_{n_{k+1}})\le\min 2^{-k-1}\{1, M_{S_{n_k}}^{-1}, \rho_{n_k}\psi_{n_k}^{-1}\|R_{n_k}\|_{X^*}^{-1}\}\omega(\psi_{n_k})$.

(16) for any $0<t\le\psi_{n_{k+1}}$ $\quad t/\omega(t)\le\min\limits_{i<k}|2^{-k-1+j}\psi_{n_j}/\omega(\psi_{n_j})\}$,

(17) $$\left|S_{n_{k+1}}\left(h_{n_{k+1}}+\sum_{j=1}^{k}\frac{\omega(\psi_{n_j})}{\omega(\psi_{n_{k+1}})}h_{n_j}\right)\right|>C_5-2^{-k-1}.$$

Define

$$f_\omega=\sum_{j=1}^{\infty}\omega(\psi_{n_j})h_{n_j}.$$

Evidently $f_\omega\in X$. We shall prove that f_ω satisfies (10—12).

For any n_0 we can find n_{j0} such that

$$\psi_{n_{j_{0+1}}}<\psi_{n_0}<\psi_{n_{j_0}}$$

therefore

$$|T_{n_0}f_\omega|<\sum_{j=1}^{j_0}\omega(\psi_{n_j})|T_{n_0}h_{n_j}|+\omega(\psi_{n_{j_0+1}})|T_{n_0}h_{n_{j_0+1}}|+\sum_{j=j_0+2}^{\infty}\omega(\psi_{n_j})|T_{n_0}h_{n_j}|=I_1+I_2+I_3.$$

Obviously from (4)

$$I_2<C_1C_2\omega(\psi_{n_{j_0+1}})<C_1C_2\omega(\psi_{n_0}),$$

using (4) and (15)

$$I_3<C_1C_2\omega(\psi_{n_{j_0+1}})\sum_{j=j_0+2}^{\infty}2^{-j}<C_9\omega(\psi_{n_0}),$$

by (5)

$$I_1<\sum_{j=1}^{j_0}\omega(\psi_{n_j})\frac{\psi_{n_0}}{\psi_{n_j}}=\omega(\psi_{n_0})\frac{\psi_{n_0}}{\psi_{n_{j_0}}}\frac{\omega(\psi_{n_{j_0}})}{\omega(\psi_{n_0})}+\omega(\psi_{n_0})\sum_{j=1}^{j_0-1}\frac{\psi_{n_0}}{\omega(\psi_{n_0})}\frac{\omega(\psi_{n_j})}{\psi_{n_j}}=J_1+J_2,$$

from (16) and $\psi_{n_0}<\psi_{n_{j_0}}$

$$J_2<\omega(\psi_{n_0})\sum_{j=1}^{j_0-1}2^{-j_0+j}<C_{10}\omega(\psi_{n_0}),$$

as for J_1, paying attention to

$$\omega(\psi_{n_{j_0}})<(\psi_{n_{j_0}}/\psi_{n_0}+1)\omega(\psi_{n_0})$$

one can get

$$J_1<2\omega(\psi_{n_0}).$$

Combine all these estimates

(18) $|T_{n_0}f_\omega|<C_6\omega(\psi_{n_0})$.

Since $\{S_n\}\subseteq X^{**}$

$$|S_{n_l}f_\omega|\geq\omega(\psi_{n_l})\left|S_{n_l}\left(h_{n_l}+\sum_{j=1}^{l-1}\frac{\omega(\psi_{n_j})}{\omega(\psi_{n_l})}h_{n_j}\right)\right|-C_1M_{S_{n_l}}\sum_{j=l+1}^{\infty}\omega(\psi_{n_j})=K_1-K_2.$$

In view of (17)

$$K_1>(C_5-2^{-l})\omega(\psi_{n_l}),$$

according to (16)

$$K_2<C_1\omega(\psi_{n_l})\sum_{j=l}^{\infty}2^{-l}$$

hence

(19) $|S_{n_l}f_\omega|/\omega(\psi_{n_l})>C_5-2^{-l}-C_1\sum_{j=l}^{\infty}2^{-j}$.

Now we turn to the estimate of $|R_{n_l}f_*|$. Notice that

$$|R_{n_l}f_*|<\sum_{j=1}^{l-1}\omega(\psi_{n_j})|R_{n_l}h_{n_j}|+\omega(\psi_{n_l})|R_{n_l}h_{n_l}|+\sum_{j=l+1}^{\infty}\omega(\psi_{n_j})|R_{n_l}h_{n_j}|=L_1+L_2+L_3,$$

from (6) and (15) respectively

$$L_2<C_4\omega(\psi_{n_l})\rho_{n_l}\psi_{n_l}^{-1},$$

$$L_3<C_1\omega(\psi_{n_l})\rho_{n_l}\psi_{n_l}^{-1}\sum_{j=1}^{\infty}2^{-j}$$

by (6) and (16)

$$L_1<C_4\sum_{j=1}^{l-1}\omega(\psi_{n_j})\rho_{n_l}\psi_{n_j}^{-1}=C_4\omega(\psi_{n_l})\rho_{n_l}\psi_{n_l}^{-1}\sum_{j=1}^{l-1}\frac{\psi_{n_l}}{\omega(\psi_{n_l})}\frac{\omega(\psi_{n_j})}{\psi_{n_j}}$$

$$<C_4\omega(\psi_{n_l})\rho_{o_l}\psi_{n_l}^{-1}\sum_{j=1}^{l-1}2^{-l+j}$$

i.e.

(20) $|R_{n_l}f_*|/(\omega(\psi_{n_l})\rho_{n_l}\psi_{n_l}^{-1})^{-1}<C_{11}$.

Combining (18)—(20), we have obtained the inequalities (10—12), thus the proof is completed.

Reference

[1] W. Dickmeis, R.J.Nessel & E. van Wickeren, A general approach to quantitative negative results in approximation theory, in "Мат структулы. Вышсл. мат. Мат. Моделир. Т. 2", София, 1984, 141—147.