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Self Complement Graphs on 8 Vertices*

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Abstract In this note we deduce that there are exactly 10 self complement graphs on 8 vertices ($G\cong\bar{G}$), which is characterized in the sort of degree sequences. It is a correction to the assertation made by Harary ([1]).

All graphs considered here are simple, finite and undirected. The terminology and notation are the same as Harary ([1]). In [1] (P.24, Ex.2.17), F. Harary made an exercise to draw the 4 self complement graphs on 8 vertices. But through the deduction as follows, we find out 10 such graphs.

Let G be a self complement graph (briefly s.c.graph) with the degree sequence $(d_1, d_2, \cdots, d_8)$, $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_8$. Let $\bar{d}_i = 7 - d_i$, thus $\bar{d}_1 \leqslant \bar{d}_2 \leqslant \cdots \leqslant \bar{d}_8$, $(\bar{d}_1, \bar{d}_2, \cdots, \bar{d}_8)$ is the degree sequence of $\bar{G}$. Since $\{d_1, d_2, \cdots, d_8\} = \{\bar{d}_1, \bar{d}_2, \cdots, \bar{d}_8\}$ by $G \cong \bar{G}$, hence $d_i = d_{9-i}$ $(i = 1, 2, \cdots, 8)$. So the degree sequence of G is determiped by the first 4 greatest integers d_1, d_2, d_3, d_4.

Proposition 1 $d_1 < 7$, $d_4 \geqslant 4$.

Proof If $d_1 = 7$, then G has a vartex v which is adjacent to to any other 7 vertices, so G is connected. But $d_1 = 0$, i.e. $\bar{G}$ has an isolated vertex, it contradicts to the fact that $\tilde{G} \cong G$ and G is connecetd, so $d_1 < 7$. If, to the opposite, $d_4 < 4$, then $d_4 = d_5 = 7 - d_4 \geqslant 4 > d_4$, a contradiction. So $d_4 \geqslant 4$.

Proposition 2 The graph as shown in Fig. 1 is the only self complement graph of order 8 with the maximum degree 6.

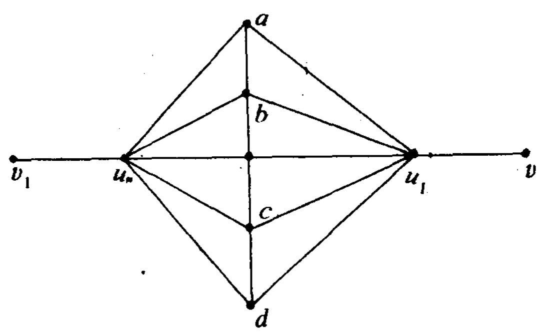

Fig.1

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Proof If $d(u)=6$, suppose v is the only vertex of G such that $(u,v)\notin E(G)$. Since $\overline{d_1}=1$, thus from $G\cong\overline{G}$, we see that G has a pendent vertex, say w, if $w=v$, then $d(u,w)=2$, thus the pair of vertices u,w in $\overline{G}$ satisfy $\overline{d}(u)=1, \overline{d}(w)=6$ and $(u,w)\in E(G)$. Since $G\cong\overline{G}$, G has another pair of adjacent vertices, say v_1 and u_1 of degree 1 and degree 6 respectively. Thus G has at least two vertices of valence 1 and, two vertices of valence 6. Let the remaining 4 vertices be a,b,c,d which join both u and u_1 and not v or v_1, so the valences of a,b,c,d are between 2 and 5. So by the isomorphic map φ from G to $\overline{G}$, $\{a,b,c,d\}$ must be mapped into $\{a,b,c,d\}$ itself. Let H be the induced subgraph of G on vertices a,b,c and d, then $H\cong\widetilde{H}$ and H must be a path, without loss of generality, say, (a,b,c,d). Then the graph we get (see Fig.1) is the only possible s.c.graph of oredr 8 with the maximum degree of 6. After an easy check we found that it is a s.c.graph. If $w\neq v$, the proof is similar to above with some slight modification. So the proposition holds.

Now $5\geqslant d_1\geqslant d_2\geqslant d_3\geqslant d_4\geqslant 4$, (d_1,d_2,d_3,d_4) can have only following 5 choices: $(5,5,5,5)$, $(5,5,5,4)$, $(5,5,4,4)$, $(5,4,4,4)$, $(4,4,4,4)$. We discuss the cases in the following propositions.

Proposition 3 The half degree sequence of G can not take the form of either $(5,5,5,4)$ or $(5,4,4,4)$.

Proof (By contradiction) If $(d_1,d_2,d_3,d_4)=(5,5,5,4)$, then the degree sequence of G is $(5,5,5,4,3,2,2,2)$. Suppose $d(u)=4$, $d(v)=3$, then by the isomorphism φ of G to $\overline{G}$, u,v is mapped into v and u respectively. We have $(z,v)\in E(G)$ if and only if $(\varphi(u),\varphi(v))\in E(G)$, i.e. $(v,u)\in E(\overline{G})$, it contradicts to the definition of $\overline{G}$. The proof of $(d_1,d_2,d_3,d_4)\neq(5,4,4,4)$ is similar to above.

Proposition 4 The graphs as shown in Fig.2 are the only two self complement graphs with the half degree sequence $(5,5,5,5)$.

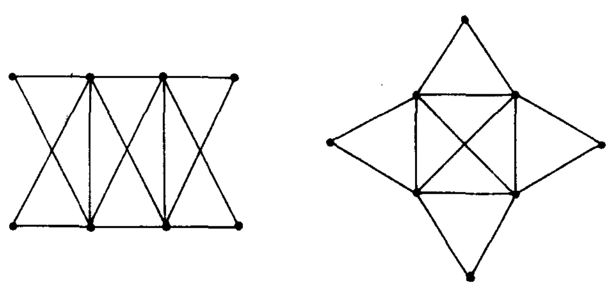

Fig.2

Proof If $(d_1,d_2,d_3,d_4)=(5,5,5,5)$, then the degree sequence of G is $(5,5,5,5,2,2,2,2)$. Let G_i be the indueced subgraph of G on the vertices of degree i. Clearly $G_5\cong\overline{G_2}$. Let $e(A,B)$ be the number of edges joining A and B. If G_5

has x edges then G_2 has $6-x$ edges and $e(G_5,G_2)=20-2x=8-2(6-x)$, henceforth $x=6$, $G_5=K_4$, $G_2=\overline{K_4}$. The graphs as shown in Fig.2 are the only two possible different such graphs (on isomorphic view), we can easily check that they are self-complemented .

Proposition 5 The graphs as shown in Fig.3 are the only three self complement graphs with the half degree sequence (55,4,4).

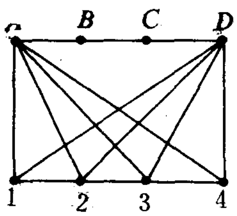

Fig.3 (1)

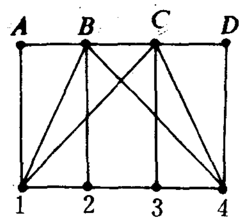

Fig.3 (2)

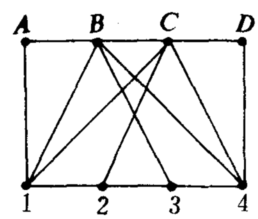

Fig.3 (3)

Proof If $(d_1,d_2,d_3,d_4)=(5,5,4,4)$, then the degree sequence of G is (5,5, 4,4,3,3,2,2). As above, $\overline{G_5}\simeq G_2$, $\overline{G_4}\cong G_3$, and $e(G_5,G_2)=e(G_4,G_3)=2$, $e(G_5\cup G_2, G_4\cup G_3)=2(4+3)-2(2+1)=8$. Since $G_5\cup G_2$ and $G_4\cup G_3$ are both s.c. graphs on four vertices, then they must be two paths. Let $G_5\cup G_2=(A, B, C, D)$, $G_4\cup G_3=(1,2,3,4)$. Since $G_5\cong\overline{G_2}$ and $G_4\cong\overline{G_3}$, so either $A, D\in G_5$ or $B, C\in G_5$; either $1,4\in G_3$ G_3 or $2,3\in G_3$.

Case 1. If $A, D\in G_5$, then both A and D are adjacent to each vertex of $G_3\cup G_4$. Then G can only be the graph as shown in Fig.3(1). After a simple check we see that G is self-complemented .

Case 2. If $B, C\in G_5$, suppose $e(G_5,G_3)=y$, $e(G_5,G_4)=z$, then $e(G_2,G_4)=4-y$, $e(G_2,G_3)=4-z$. Clearly $y+z=6$.

Subcase 1. If $2,3\in G_4$, then $y+(4-z)=4$, i.e. $y=z=3$, $e(G_2,G_4)=e(G_2,G_3)=1$, without loss of generality, suppose $(4,D)\in E(G)$. Let $(i,A)\in E(G)$ (where $i=2$ or 3), then the remaining two vertices j,k of $\{1,2,3\}$ are adjacent to both B and C. This time, j,k can not join either B or C in $\overline{G}$. So again only 4 and i join B and C of degree 2 in $\overline{G}$. Thus $\varphi\{4,i\}=\{4,i\}$, it is impossible. We deduce that in this case, the s.c. graph is inexistent.

Subcase 2. If $2,2\notin G_3$, then $y+z=6$, $(4-y)+z=6$ i.e. $y=2$, $z=4$, $e(G_2G_4)=e(G_3,G_5)=2$, $e(G_4,G_5)=4$, $e(G_2,G_3)=0$. Since $d(A)=d(D)=2$, $d(1)=d(4)=4$. So 1 and 4 join both B and C. Thus, without loss of generality, we assign $(1,A)$, $(4,D)\in E(G)$. Now we have two choices: either $(2,B)$, $(3,C)\in E(G)$ or $(2,C)$, $(3,B)\in E(G)$. In fact the two graphs shown in Fig.3 (2), (3) are s.c.graphs.

Proposition 6 The graphs as shown in Fig.4 are the only four self complement graphs with the half degree suquence (4,4,4,4).

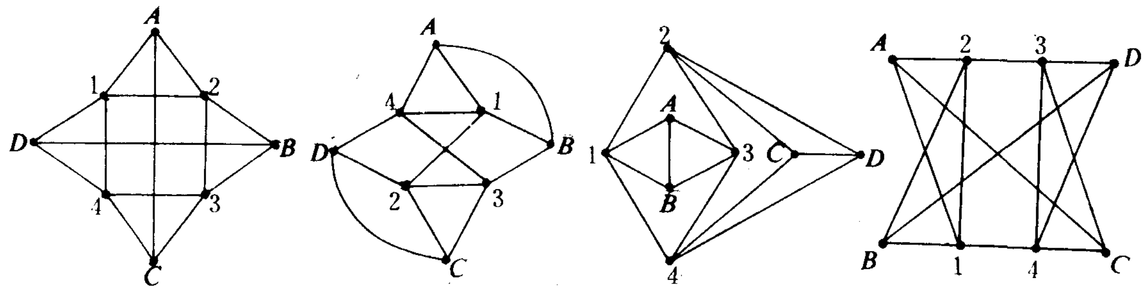

Fig.4 (1) Fig.4 (2) Fig.4 (3) Fig.4 (4)

Proof If G is a s.c. graph with the degree sequence (4,4,4,4,3,3,3,3), then $\overline{G_4} \cong G_3$ and $e(G_4, G_3) = 8$. Suppose G_4 has x edges then G_3 has $6-x$ edges. Hence $e(G_4, G_3) = 16 - 2x = 8$, thus $x = 4$, G_4 is either a cycle or a "handle" as shown in Fig5. (1).

Fig.5 (1) Fig.5 (2)

Cese 1. If G_4 is a cycle, say (1,2,3,4,1), then G_3 must be composed of two independent edges, say e_1 and e_2. Not considering e_1 and e_2, we have four possible subcases as follows:

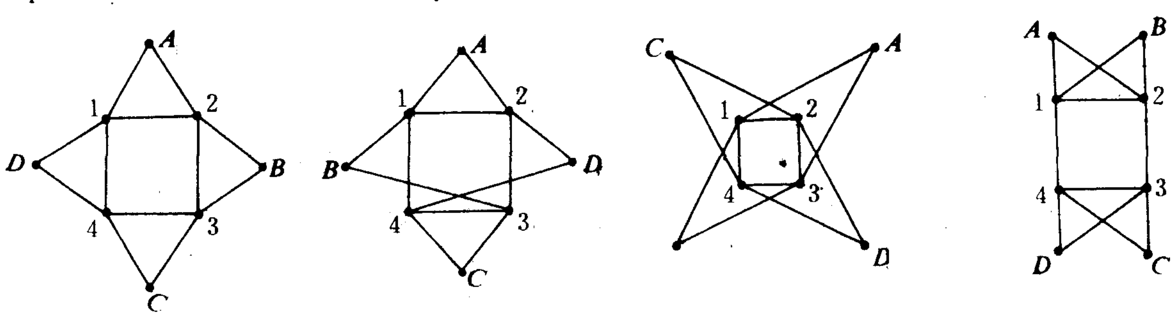

Fig.6 (1) Fig.6 (2) Fig.6 (3) Fig.6 (4)

Subcase 1: Suppose $V(G_3) = \{A, B, C, D\}$, A join 1 and 2, B join 2 and 3, C join 3 and 4 and D join 4 and 1. For any vertex of valence 3, its two neighbnur vertices of valence 4 are adjacent in G. Since $G \cong \overline{G}$, $\overline{G}$ also has this property. Clearly A, B, C, D bear the degree 4 in $\overline{G}$, and A, B are the neighbour of vertex 4 which bear the degree 3. Thus $(A, B) \in E(\overline{G})$. Similarly (B, C), (C, D), $(A, D) \in$

$E(\overline{G})$, i.e. (A, B), (B, C), (C, D), $(D, A) \notin E(G)$. Hence it is only possible that (A, C), $(B, D) \in E(G)$. We get a graph as shown in Fig.4 (1). It is easy to check that the graph is self-complemented.

Subcase 2: Suppose A join 1 and 2, C join 3 and 4, B join 1 and 3, D join 2 and 4. As we can see, G has a vertex of valence 3 (say B) whose neighbour vertices of valence 4 (1 and 3) are not abjacent. Since $G \cong \overline{G}$, $\overline{G}$ also has this property, that is to say, G has two adjacent vertices of valence 3 which are both adjacent to a same vertex of valence 4. Without loss of generality, suppose (A, B), $(C, D) \in E(G)$. We get a graph as shown in Fig.4 (2) and it is a s.c. graph after an easy check.

Subcase 3: Suppose both A and B join 1 and 3, C and D join 2 and 4. Similar to above we found that it is only possible that (A, B), $(C, D) \in E(G)$. We get a graph as shown in Fig.4 (3) and it is actually a s.c. graph.

Subcase 4: Suppose both A and B join 1 and 2, C and D join 3 and 4. Consider vertices 1 and 2, we see that G has two adjacent vertices of valence 4 such that neither of which join some two vertices of valence 3(i.e. C and D). From $G \cong \overline{G}$, $\overline{G}$ also has this property. It is to say that G has two nonadjacent vertices of valence 3 such that both of which join some two vertices of valence 4. So (A, B), $(C, D) \notin E(G)$. Thus, without loss of generaltiy, we say (A, C), , $(B, D) \in E(G)$. We get a s.c.graph as shown in Fig.4 (4).

Case 2 If G_4 is a "handle" we will show that such s.c.graph do not exist. If otherwise, G is a s.c. graph with G_4 being a "handle", then G_3 is the graph as shown in Fig.5 (2). Since G has only one vertex A of valence 4 which join exactly one vertex of valence 3. From $G \cong \overline{G}$, we see $\overline{G}$ also has this property, so $\varphi(A) = a$. Similarly, $\varphi(a) = A$, a contradiction.

Remark In a forthcoming paper, the author characterize the s.c. graphs on 9 vertices and furthermore from these results an algorithm is made to construct all the s.c. graphs on $4n$ or $4n+1$ vertices.

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Reference

[1] F. Harary. Graph Theory, Addison wesley, Reading MA (1969).