Journal of Mathematical Research and Exposition
Vol. 10, No.2, May, 1990

Existence and Uniqueness of Fixed Point for A Mapping On Product Banach Space And Convergence of Iterations*

You Zhaoyong Li Lei

(Xi'an Jiaotong University)

It is well-known that when Newton method, steepest descent method and other iterative method on non-linear equation are generalized to system of equations, their convergence conditions are very complex (e.g., computing of Jacobi matrix), and construction of large range convergence conditions are more difficult. Although Banach contraction mapping principle is usually used to solve various function equations, sometimes it is difficult to judge whether there exists a Lipschitz constant which is less than 1. In [1] Seidel iteration method for bounded linear operator in Hilbert space and its convergence criterion are given, but they are not practical because the assumption is very strong. In [2] existence and uniqueness for some non-linear vector operators are shown by the generalization of [3] and convergence criterions are given, which are practical and simple.

In this paper, existence and uniqueness of fixed point for non-linear vector operator from product space of n Banach spaces to itself are shown by generalization of convergence criterions of Jacobi iteration and Seidel iteration for linear mapping from R^n to itself in [4], and, hence, main results of [2] are generalized. Banach contraction mapping principle, as a most special case, is contained by all conclusions in this paper.

Let

$$B = B_1 \times B_2 \times \cdots \times B_n$$

where B_i is complex Banach space and operator

$$T = (T_1, T_2, \cdots, T_n): B \to B$$

Consider system of equations

$$x_i = T_i(x_1, x_2, \cdots, x_n), \quad x_i \in B_i, \ i = 1, \cdots, n \tag{1}$$

or $x = Tx$ for short.

First, consider Seidel iteration

$$x_i^{(m)} = T_i(x_1^{(m)}, x_2^{(m)}, \cdots, x_{i-1}^{(m)}, x_i^{(m-1)}, \cdots, x_n^{(m-1)}) \qquad i = 1, \cdots, n \tag{2}$$

* Received Jul. 23, 1987

Theorem 1 Suppose that $T: G \subset B \to B$ maps a closed set $G_0 \subset G$ into itself and that .

1°. T is a Lipschitz continuous operator, namely,

$$\|T_i(x_1, \cdots, x_n) - T_i(y_1, \cdots, y_n)\| < \sum_{j=1}^{n} \rho_{ij} \|x_j - y_j\| \qquad (3)$$

2°. $I-P$ is a generalized diagonally dominant matrix whose diagonal elements are positive, i.e., there exist positive constants d_i, $i = 1, \cdots, n$, such that

$$(1 - \rho_{ii}) d_i > \sum_{j \neq i} \rho_{ij} d_j \quad, i = 1, 2, \cdots, n \qquad (4)$$

where I is a unit matrix and $P = (\rho_{ij})_{n \times n}$. Then T has a unique fixed point $x^* \in G_0$, and for any $x^0 \in G_0$ the sequence (2) converges to x^*. (acturally, the condition 2° is equivalent to that $I-P$ is an M-matrix).

Proof From assumption, there exist positive constants d_i, $i = 1, \cdots, n$, such that

$$(1 - \rho_{ii}) d_i > \sum_{j \neq i} \rho_{ij} d_j \qquad i = 1, \cdots, n$$

Define a homemorphism: $x_i = d_i y_i$, $i = 1, \cdots, n$, then (1) become as

$$d_i y_i = T_i(d_1 y_1, d_2 y_2, \cdots, d_n y_n)$$

or

$$y_i = d_i^{-1} T_i(d_1 y_1, d_2 y_2, \cdots, d_n y_n) \overset{\text{def}}{=} T_i'(y_1, y_2, \cdots, y_n) \quad i = 1, \cdots, n \qquad (5)$$

Obviously, T' maps the closed set $d^{-1} G_0$ into itself, and

$$\begin{aligned} &\|T_i'(s_1, s_2, \cdots, s_n) - T_i'(t_1, t_2, \cdots, t_n)\| \\ &= \|d_i^{-1} T_i(d_1 s_1, d_2 s_2, \cdots, d_n s_n) - d_i^{-1} T_i(d_1 t_1, d_2 t_2, \cdots, d_n t_n)\| \\ &< d_i^{-1} \sum_{j=1}^{n} \rho_{ij} \|d_j s_j - d_j t_j\| < d_i^{-1} \sum_{j=1}^{n} \rho_{ij} d_j \|s_j - t_j\| \end{aligned}$$

Let

$$\rho_{ij}' = d_i^{-1} \rho_{ij} d_j, \qquad i, j = 1, \cdots, n$$

then ρ_{ij}' is a Lipschitz constant of (5), and hence

$$\|T_i'(s_1, s_2, \cdots, s_n) - T_i'(t_1, t_2, \cdots, t_n)\| < \sum_{j=1}^{n} \rho_{ij}' \|s_j - t_j\|.$$

From (4), we can get

$$\sum_{j=1}^{n} \rho_{ij}' < 1, \qquad i = 1, \cdots, n$$

According to Theorem 1 in [2], T' has a unique fixed point $y^* \in d^{-1} G_0$, and the sequence (5) converges to y^*, and so that T has a unique fixed point $x^* \in G_0$ and Seidel iteration (2) for (1) converges to x^* (in fact, we can show that $d_i y_i^{(m)} = x_i^{(m)}$, $i = 1, \cdots, n$, $m = 1, 2, \cdots$, $d_i y_i^* = x_i^*$, $i = 1, \cdots, n$). ■

Above mentioned proof process can be illustrated by following figure:

$$G_0 \quad \begin{matrix} x^{(0)} \\ \vdots \\ x^{*} \end{matrix} \qquad\qquad d^{-1}G_0 \quad \begin{matrix} y^{(0)} \\ \vdots \\ y^{(*)} \end{matrix}$$

We have shown the existence and uniqueness of fixed point in complex Banach space. Next, we will give a direct proof on the uniqueness of fixed point in general Banach space.

Suppose x_i^* and $\tilde{x}$ be two fixed points of T, then

$$\|x_i^* - \tilde{x}_i\| = T_i(x_1^*, \cdots, x_n^*) - T_i(\tilde{x}_1, \cdots, \tilde{x}_n)\| < \sum_{j=1}^{n} \rho_{ij}\|x_j^* - \tilde{x}_j\| \quad i = 1, \cdots, n$$

Let

$$\|x^* - \tilde{x}\| = (\|x_1^* - \tilde{x}_1\|, \cdots, \|x_n^* - \tilde{x}_n\|)^T$$

then

$$\|x^* - \tilde{x}\| < P\|x^* - \tilde{x}\|$$

or

$$(I - P)\|x^* - \tilde{x}\| < 0$$

According to the assumption 2°, clearly, $I - P$ is an M-matrix, i.e., $(I - P)$ is a non-negative matrix, and hence,

$$\|x^* - \tilde{x}\| < 0$$

which implies $\|x^* - \tilde{x}\| = 0$, or $x^* = \tilde{x}$. ■

Notes We point out that Theorem 1 is equivalent to [7] in theory, but their proof is different.

Next, let us consider Jacobi iteration

$$x_i^{(m)} = T_i(x_1^{(m-1)}, x_2^{(m-1)}, \cdots, x_n^{(m-1)})$$

or

$$x^{(m)} = T(x^{(m-1)}) \tag{6}$$

for short.

Theorem 2 Let G be an open set. Suppose that $T: G \subset B \to B$ maps a closed set $G_0 \subset G$ into itself and that

1°. the condition 1° of Theorem 1,

2°. the condition 2° of Theorem 1.

Then T has a unique fixed point $x^* \in G_0$, and for any $x^{(0)} \in G_0$ the sequence (6) converges to x^*.

Proof When T satisfies the condition 1° and 2°, we have shown that T has a unique fixed point in G_0(Theorem 1), and so we only show sequence (6) converges to x^*. In fact, according to proof of Theorem 1, define a homeomorphism: $x_i = d_i y_i$, $i = 1, \cdots, n$, then Lipschitz constant ρ'_{ij} of (5) satisfies that $\sum_{j=1}^{n} \rho'_{ij} < 1$, $i = 1, \cdots, n$. According to Theorem 4 in [2]. Jacobi iterative sequence for (5) converges to y^*, and thus, sequence (6) on variable x_i also converges to, x^*. (see Theorem 12.1.5 and 12.1.6 in [5]). ■

According to some sufficient conditions for generalized diagonally dominant matrix in [4], we can get practical convergence criterions for Seidel iteration and Jacobi iteration of function equation (1). These results contain all related conclusions in [2], especially, Banach contraction mapping principle.

Theorem 3 Let G be an open set. Suppose that $T: G\subset B\to B$ maps a closed set $G_0\subset G$ into itself and that

1°. the condition 1° of Theorem 1

2°. $\rho_{ii}<1$, $i=1,\cdots,n$, and one of the following conditions is satisfied

a) $\sum_{j=2}^{n}\frac{\rho_{1j}}{1-\rho_{11}}=u_1<1,\quad \frac{\rho_{21}}{1-\rho_{22}}u_1+\sum_{k=3}^{n}\frac{\rho_{2j}}{1-\rho_{22}}=u_2<1,\cdots,\sum_{j=1}^{n-1}\frac{\rho_{nj}}{1-\rho_{nn}}u_j=u_n<1.$

b) $(1-\rho_{ii})(1-\rho_{jj})>\sum_{k\neq i}\rho_{ik}\sum_{j\neq j}\rho_{jk},\qquad i\neq j$

c) $\sum_{k=1}^{n}\frac{a^{(k)}}{1+a^{(k)}}=a<1,$ where

$$a^{(1)}=\max_{j\neq 1}\frac{\rho_{1j}}{1-\rho_{11}},\quad a^{(2)}=\max_{j\neq 2}\frac{\rho_{2j}}{1-\rho_{22}},\cdots,\quad a^{(n)}=\max_{j\neq n}\frac{\rho_{nj}}{1-\rho_{nn}}.$$

d) $\sum_{k=1}^{n}b^{(k)}=b<1,$ where

$$b^{(1)}=\max_{1\leq k\leq n}\rho_{1k},\quad b^{(2)}=\rho_{21}b^{(1)}\neq\max_{2\leq k\leq n}\rho_{2k},\cdots,b^{(n)}=\sum_{k=1}^{n-1}\rho_{nk}b^{(k)}+\rho_{nn}$$

e) $\sum_{k=1}^{n}a^{(k)}=a<1,$ where

$$a^{(1)}=\max_{2\leq k\leq n}\frac{\rho_{1k}}{1-\rho_{11}},\quad a^{(2)}=\frac{\rho_{21}}{1-\rho_{22}}a^{(1)}+\max_{3\leq k\leq n}\frac{\rho_{2k}}{1-\rho_{22}},\cdots,$$

$$a^{(n)}=\sum_{k=1}^{n-1}\frac{\rho_{nk}}{1-\rho_{nn}}a^{(k)}$$

f) $\sum_{k=1}^{n}a^{(k)}=a<1,$ where

$$a^{(1)}=\frac{r_1}{1+r_1},\quad a^{(2)}=[(\frac{\rho_{21}}{1-\rho_{22}}-r_2)a^{(1)}+r_2]/(1+r_2),$$

$$a^{(3)}=[(\frac{\rho_{31}}{1-\rho_{33}}-r_3)a^{(1)}+(\frac{\rho_{32}}{1-\rho_{33}}-r_3)a^{(2)}+r_3]/(1+r_3),\cdots,$$

$$a^{(n)}=\sum_{k=1}^{n-1}\frac{\rho_{nk}}{1-\rho_{nn}}a^{(k)},\ r_k=\max_{k+1\leq j\leq n}\frac{\rho_{kj}}{1-\rho_{kk}}.$$

g) $\sum_{i,j=1}^{n}\rho_{ij}^2<1$

h) $\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{\rho_{ij}}{1-\rho_{ii}}\right)^2 < 1$

i) $\sum_{i=1}^{n} \sigma_i^2 \triangleq \sigma^2 < 1$, where

$$\sum_{j=2}^{n}\left(\frac{\rho_{1j}}{1-\rho_{11}}\right)^2 \triangleq \sigma^1,\ 2\left[\left(\frac{\rho_{21}}{1-\rho_{22}}\right)^2\sigma_1^2 + \sum_{j=3}^{n}\left(\frac{\rho_{2j}}{1-\rho_{22}}\right)^2\right] \triangleq \sigma_2^2,\ \cdots,$$

$$n\left[\sum_{j=1}^{n-1}\left(\frac{\rho_{nj}}{1-\rho_{nn}}\right)^2\sigma_j^2\right] \triangleq \sigma_n^2$$

Then T has a unique fixed point $x^* \in G_0$, and for any $x^{(0)} \in G$, Seidel iteration and Jacobi iteration all converge to x^*.

Notes Iteration convergence depends on the order of $T_1, T_2, \cdots, T_n$ and $B_1, B_2, \cdots, B_n$. If condition a)~i) is not satisfied, we can make them be satisfied by adjusting the order of $T_i, B_i, i=1,\cdots,n$, properly, and hence we can extend applying range (see [3] and [4]).

Corollary 1 When $n=1$, $B_1=B$, $x_1=x$, $x_1=T(x_1)$, i.e., $x=T(x)$ and $0<\rho_{11}=\rho<1$, we can get Banach contraction mapping priciple.

Corollary 2 Let $T: G \subset B \to B$ be a differentiable operator whose Frechet partial derivative operator satisfies that

1° $\sup_{x \in G} \left\| \frac{\partial T_i}{\partial x_j} \right\| < \rho_{ij}$,

2° The condition 2° of Theorem 1.

Then conclusions of Theorem 1 and Theorem 2 all hold.

Finally, we point out that main conclusions of Theorem 1~4 in [2] are all contained by Theorem 3 in this paper.

Reference

[1] Guan Zhaozhi, Functional Analysis, Higher Education Press, 1958.

[2] Liao Xiaoxin, Mathematica Numerica Sinica, 6:4(1984), 337—350.

[3] Liao Xiaoxin, Mathematica Numerica Sinica 1:2(1979), 164—171.

[4] Li Lei, Convergence Criterions (I), (II) and (III) For Arbitrary Splitting Form, submitting to Mathematica Numerica Sinica.

[5] Ortega, J. M., and Rheinboldt, W. C. Iterative Solution of Nonlinear Equations in Several Variables Academic Press, New York, 1970.

[6] You Zhaoyong, Nonsingular M-matrix, Huazhong Universsity, 1981.

[7] Liao Xiaoxin, Science Bulletin, No. 12(1883).

Banach乘积空间中映射的不动点存在唯一性与迭代过程的收敛性

游 兆 永　李 磊

(西安交通大学应用数学系)

摘　　要

本文将作者在[4]中得到的从R^n到R^n的线性映射下Jacobi迭代和Seidel迭代收敛准则推广到论证n个Banach空间的乘积空间映射到自良的非线性向量算子的不动点存在唯一性，推广了[2]的主要结论．本文所有结论均包含Banach压缩映射原理作为最简单的特例．

接220页

参 考 文 献

[1] S. Reich, A remark on set-valued mapping that Satisfy the Leery-Schauder condition, Atti. Accad. Naz. Lincei .61(1976)193—194.

[2] S. Reich, A remark on set-valued mapping that satisfy the Leary-Schauder Condition, Atti. Accad. Naz. Lincei. 66(1979) 1—2.

[3] M. Lassonde, On the use of KKM multifunction in fixed point theory and related topics, J. Math. Anal. Appl. 97(1983) 151—201.