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On Multivalent Functions with Negative and Missing Coefficients*

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Abstract Let $P_k(p, A, B)$ be the class of functions $f(z)=z^p-\sum_{n=k}^{\infty}|a_{n+p}|z^{n+p}$ $k\geq 2$ analytic in the unit disc $E=\{z: |z|<1\}$ and satisfying the condition $|(zf'(z)/f(z)-p)/(Ap-Bzf'(z)/f(z))|<1$. for $z\in E$ and $-1\leq B<A\leq 1$. In this paper we obtain representation formula, coefficient estimate, distortion and closure theorems and the radius of convexity for the class $P_k(p, A, B)$ under the assumption $-1\leq B<0$.

I. Introuction

Let $S(p)$ be the class of functions $f(z)=z^p+\sum_{n=z}^{\infty}a_{n+p}z^{n+p}$ which are analytic in the unit disc $E=\{z: |z|<1\}$. For $-1\leq B<A\leq 1$ let $P^*(p, A, B)$ be the class of those functions f of $S(p)$ which satisfy the condition

$$|(zf'(z)/f(z)-p)/(Ap-Bzf'(z)/f(z)|<1 \quad \text{for } z\in E \tag{1}$$

Let T_p denote the sub-class of $S(p)$ consisting of p-valent functions in E and having Taylor expansion of the form

$$f(z)=z^p-\sum_{n=k}^{\infty}|a_{n+p}|z^{n+p}, \quad k\geq 2.$$

Let $P_k(p, A, B)=P^*(p, A, B)\cap T_p$

Goel and Sohi [1], Sarangi and Uralegaddi [2], Shukla and Dashrath [3], Herb Silverman [4] have studied certain sub-classes of analytic functions with negative coefficients and Vinod Kumar [5] has recently studied the class of univalent functions with negative and missing coefficients.

In this paper, under the assumption $-1\leq B<0$, and $k\geq 2$, we obtain representation formula, coefficient estimate, distortion theorem, covering theorem and radius of convexity for $P_k(p, A, B)$

We also obtain the class preserving integral operators of the form

$$F(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)\,dt, \quad c>-1 \tag{2}$$

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for $P_k(p, A, B)$. Conversely wher $F \in P_k(p, A, B)$ we determine the radiue of p-valence of f defined by (2). Lastly we show that the class $P_k(p, A, B)$ is closed under "Arithmetic mean" and "convex linear combinations".

2. Representation Formula

Theorem 1 The function $f(z) = z^p - \sum_{n=k}^{\infty} |a_{n+p}| z^{n+p}$ belongs to $P_k(p, A, B)$ if and only if it can be expressed in the form

$$f(z) = z^p \exp\left[p(A-B)\int_0^z \frac{t^{k-1}\phi(t)}{1+Bt^k\phi(t)} dt \right]$$

where $\phi(z)$ is analytic in E and satisfies $|\phi(z)| \leq 1$ for $z \in E$.

Proof Let $f(z) \in P_k(p, A, B)$. Then

$$|(zf'(z)/f(z) - p)/(Ap - Bzf(z)/f(z)| < 1 \text{ for } z \in E$$

and since the absolute value vanishes for $z = 0$, we have

$$(zf(z)/f(z) - p)/(Ap - Bzf'(z)/f(z)) = z^k\phi(z) \tag{1}$$

where $\phi(z)$ is analytic function in E and satisfies $|\phi(z)| \leq 1$ for $z \in E$. $(zf'(z)/f(z) - p)/(Ap - Bzf'(z)/f(z)) = z^k\phi(z)$ implies

$$\frac{1}{p(A-B)}\left[\frac{z^{-p}f(z) - pz^{-p-1}f(z)}{z^{-p}f(z)}\right] = \frac{z^{k-1}\phi(z)}{1+Bz^k\phi(z)}$$

which on integrating and simplifying gives

$$f(z) = z^p \exp\left[p(A-B)\int_0^z \frac{t^{k-1}\phi(t)}{1+Bt^k\phi(t)} dt \right]$$

conversly suppose

$$f(z) = z^p \exp\left[p(A-B)\int_0^z \frac{t^{k-1}\phi(t)}{1+Bt^k\phi(t)} dt \right] \tag{2}$$

implies $\log z^{-p} \cdot f(z) = p(A-B)\int_0^z \frac{t^{k-1}\phi(t)}{1+Bt^k\phi(t)} dt$

So differentiating and simplifying we get

$$\frac{1}{p(A-B)}\left[\frac{zf'(z)}{f(z)} - p\right] = \frac{z^k\phi(z)}{1+Bz^k\phi(z)} \tag{3}$$

$$\left|\frac{zf'(z)}{f(z)} - p\right| = p(A-B)\frac{|z^k\phi(z)|}{|1+Bz^k\phi(z)|} < \frac{p(A-B)}{|1+Bz^k\phi(z)|}.$$

Since $|z^k\phi(z)| < 1$.

Substituting $z^k\phi(z)$ from (1) and simplyfying we get

$$|(zf'(z)/f(z) - p)/(Ap - Bzf'(z)/f(z))| < 1.$$

Hence $f(z) \in P_k(p, A, B)$.

3. Coefficient Estimate

Theorem 2 A function $f(z)=z^p-\sum_{n=k}^{\infty}|a_{n+p}|z^{n+p}$ is in $P_k(p, A, B)$ if and only if

$$\sum_{n=k}^{\infty}[(1-B)n+(A-B)p]\,|a_{n+p}|\leq (A-B)p.$$

Proof Suppose $\sum_{n=k}^{\infty}[(1-B)n+(A-B)p]\,|a_{n+p}|\leq (A-B)p$ is true. Then

$$\left|\frac{zf'(z)}{f(z)}-p\right|-\left|Ap-B\frac{zf'(z)}{f(z)}\right|<0$$

provided

$$\left|-\sum_{n=k}^{\infty}n\,a_{n+p}z^{n+p}\right|-\left|(A-B)pz^p+(B-A)p\sum_{n=k}^{\infty}|a_{n+p}|z^{n+p}+B\sum_{n+k}^{\infty}n\,|a_{n+p}|z^{n+p}\right|<0$$

For $z=r<1$ the left hand side of the above inequality is bounded above by

$$\sum_{n=k}^{\infty}n\,|a_{n+p}|r^{n+p}-(B-A)p\sum_{n=k}^{\infty}|a_{n+p}|r^{n+p}-B\sum_{n=k}^{\infty}n\,|a_{n+p}|r^{n+p}-(A-B)pr^p$$

$$=\sum_{n=k}^{\infty}[(1-B)n+(A-B)p]\,|a_{n+p}|r^{n+p}-(A-B)pr^p$$

$$<\sum_{n=k}^{\infty}[(1-B)n+(A-B)p]\,|a_{n+p}|-(A-B)p<0\,.$$

Hence $f(z)\in P_k(p, A, B)$.

Conversely suppose that $f(z)\in P_k(p, A, B)$ then

$$|(zf'(z)/f(z)-p)/(Ap-Bzf'(z)/f(z))|$$

$$=\left|\frac{-\sum_{n=k}^{\infty}n\,|a_{n+p}|z^{n+}\ \sum_{n=k}^{\infty}n\,|a_{n+p}|z^{n+}}{(A-B)pz^p+\sum_{n=k}^{\infty}[(B-A)p+Bn]\,|a_{n+p}|z^{n+p}}\right|<1 \quad \text{for } z\in E \qquad (1)$$

Since $\mathrm{Re}(z)\leq|z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum_{n=k}^{\infty}n\,|a_{n+p}|z^{n+p}}{(A-B)pz^p+\sum_{n=k}^{\infty}[(B-A)p+Bn]\,|a_{n+p}|z^{n+p}}\right\}<1 \qquad (2)$$

Choose values of z on real axis so that $\frac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator of (2) and letting $z\to 1$ through real values, have

$$\sum_{n=k}^{\infty}n\,|a_{n+p}|\leq (A-B)p+\sum_{n+k}^{\infty}[(B-A)p+Bn]\,|a_{n+p}|,$$

So $\sum_{n=k}^{\infty}[(1-B)n+(A-B)p]\,|a_{n+p}|\leq (A-B)p$. The function

$$f(z)=z^p-\sum_{n=k}^{\infty}\frac{(A-B)pz^{n+p}}{[(1-B)n+(A-B)p]}$$

is an extremal function.

4. Distortion Properties

Theorem 3 If $f(z)\in P_k(p,A,B)$ then for $|z|=r$

$$r^p-\frac{(A-B)pr^{p+k}}{[(1-B)+(A-B)p]}\le|f(z)|\le r^p+\frac{(A-B)pr^{p+k}}{[(1-B)+(A-B)p]}$$

$$pr^{p-1}-\frac{p(p+1)(A-B)r^{n+k-1}}{[(1-B)+(A-B)p]}\le|f'(z)|\le pr^{p-1}+\frac{p(p+1)(A+B)r^{n+k-1}}{[(1-B)+(A+B)p]}.$$

Proof From theorem (2) we have

$$\sum_{n=k}^{\infty}[(1-B)n+(A-B)p]\,|a_{n+p}|<(A-B)p. \tag{1}$$

Now since $(1-B)n>(1-B)$ we have

$$[(1-B)+(A-B)p]\sum_{n=k}^{\infty}|a_{n+p}|\le\sum_{n=k}^{\infty}[(1-B)n+(A-B)p]\,|a_{n+p}|\le(A-B)p$$

So

$$\sum_{n=k}^{\infty}|a_{n+p}|\le\frac{(A-B)p}{[(1-B)+(A-B)p]}. \tag{2}$$

Now we have

$$|f(z)|=|z^p-\sum_{n=k}^{\infty}|a_{n+p}|z^{n+p}|\le r^p+\frac{(A-B)pr^{k+p}}{[(1-B)+(A-B)p]} \tag{3}$$

and

$$|f(z)|\ge|z^p|-\sum_{n=k}^{\infty}|a_{n+p}|\,|z|^{n+p}=r^p-\frac{(A-B)pr^{k+p}}{[(1-B)+(A-B)p]}. \tag{4}$$

From results (3) and (4) we have

$$r^p-\frac{(A-B)pr^{r+p}}{[(1-B)+(A-B)p]}\le r^p+\frac{(A-B)pr^{k+p}}{[(1-B)+(A-B)p]}. \tag{x}$$

Further

$$|f'(z)|\le pr^{p-1}+\sum_{n=k}^{\infty}(n+p)\,|a_{n+p}|r^{n+p-1} \tag{5}$$

and

$$|f'(z)|\le pr^{p-1}-\sum_{n=k}^{\infty}(n+p)\,|a_{n+p}|r^{n+p-1}. \tag{6}$$

Using the result $\sum_{n=k}^{\infty}|a_{n+p}|\le\frac{(A-B)p}{[(1-B)+(A-B)p]}$ in the result $\sum_{n=k}^{\infty}[(1-B)n+(A-B)p]\,|a_{n+p}|\le(A-B)p$ of theorem 2 and simplifying we have

$$\sum_{n=k}^{\infty}(n+p)\,|a_{n+p}|\leq\frac{p(p+1)(A-B)}{[(1-B)+(A-B)p]}.$$

Substituting this value of $\sum_{n=k}^{\infty}(n+p)\,|a_{n+p}|$ in (5) and (6) we have

$$pr^{p-1}-\frac{p(p+1)(A-B)r^{k+p-1}}{(1-B)+(A-B)p}\leq|f'(z)|$$

$$\leq pr^{p-1}+\frac{p(p+1)(A-B)r^{k+p-1}}{(1-B)+(A-B)p}. \qquad (y)$$

Equality in (x) and (y) is obtained if we take

$$f(z)=z^{p}-\frac{(A+B)pz^{k+p}}{[(1-B)+(A-B)p]}.$$

Corollary If $f\in P_k(p, A, B)$ then the disc E is mapped by f onto a domain that contains the disc $|w|<\frac{1-B}{(1-B)+(A-B)p}$. The result is sharp with extremal function $f(z)=\frac{(A-B)pz^{k+p}}{(1-B)+(A-B)p}$

Proof By letting $r\to 1$ in the L.H.S. of inequality (x) we have

$$1-\frac{(A-B)p}{[(1-B)+(A-B)p]}\leq|f(z)|$$

hence

$$\frac{1-B}{(1-B)+(A-B)p}\leq|f(z)|$$

so f maps the disc E onto a domain that contains the disc

$$|w|<\frac{1-B}{[(1-B)+(A-B)p]}.$$

5. Integral Operators

Theorem 4 Let c be a real number such that $c>-1$. If $f\in P_k(p, A, B)$ then the function F defined by $F(z)=\frac{c+p}{z^{c}}\int_0^z t^{c-1}f(t)\,dt$ also belongs to $P_k(p, A, B)$.

Proof Let $f(z)=z^{p}-\sum_{n=k}^{\infty}|a_{n+p}|z^{n+p}$. Then

$$F(z)=\frac{c+p}{z^{c}}\int_0^z t^{c-1}f(t)\,dt,$$

implies

$$F(z)=z^{p}-\sum_{n=k}^{\infty}\frac{p+c}{n+p+c}|a_{n+p}|z^{n+p}=z^{p}-\sum_{n=k}^{\infty}|b_{n+p}|z^{n+p},$$

where $\quad |b_{n+p}|=\frac{(p+c)}{n+p+c}|a_{n+p}|.$

Therefore using theorem (2) for the coefficients of $F(z)$ we have

$$\sum_{n=k}^{\infty}[(1-B)n+(A-B)p]|b_{n+p}|=\sum_{n=k}^{\infty}[(1-B)n+(A-B)p](\frac{c+p}{n+p+c})|a_{n+p}|$$

$\leq(A-B)p$ since $\frac{c+p}{n+c+p}<1$ and $f\in P_k(p, A, B)$.

Hence $F\in P_k(p, A, B)$.

6. Radius of Convexity

Theorem 5 If $f(z)\in P_k(p, A, B)$, then $f(z)$ is p-valently convex in the disc $|z|<R_p$, where

$$R_p=\inf_{n\geq k}\{[\frac{(1-B)n+(A-B)p}{(A-B)p}](\frac{p}{n+p})^2\}^{\frac{1}{n}} \text{ for } |z|<R_p.$$

Proof It is sufficient to show that $\left|(1+\frac{zf''(z)}{f'(z)})-p\right|\leq p$ for $|z|<R_p$.

Now $f(z)=z^p-\sum_{n=k}^{\infty}|a_{n+p}|z^{n+p}$. So

$$1+\frac{zf''(z)}{f'(z)}-p=\frac{-\sum_{n=k}^{\infty}n(n+p)|a_{n+p}|z^n}{p-\sum_{n=k}^{\infty}(n+p)|a_{n+p}|z_n}.$$

Therefore

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|\leq p \text{ if } \frac{\sum_{n=k}^{\infty}n(n+p)|a_{n+p}||z|^n}{p-\sum_{n=k}^{\infty}(n+p)|a_{n+p}||z|^n}\leq p$$

or

$$\sum_{n=k}^{\infty}(\frac{n+p}{p})^2|a_{n+p}||z|^n\leq 1. \qquad (1)$$

From theorem 2 we have

$$\sum_{n=k}^{\infty}[\frac{(1-B)n+(A-B)p}{(A-B)p}]|a_{n+p}|\leq 1.$$

Hence (1) will be satisfied if

$$(\frac{n+p}{p})^2|z|^n\leq[\frac{(1-B)n+(A-B)p}{(A-B)p}]$$

or if

$$|z|\leq[\{\frac{(1-B)n+(A-B)p}{(A-B)p}\}(\frac{p}{n+p})^2]^{\frac{1}{n}}.$$

So $f(z)$ is p-valently convex in the disc

$$|z|<R_p=\inf_{n\geq k}\left[\left\{\frac{(1-B)n+(A-B)p}{(A-B)p}\right\}\left(\frac{p}{n+p}\right)^2\right]^{\frac{1}{n}}.$$

7. Closure Properties

In this section, we show that the class $P_k(p, A, B)$ is closed under 'Arithmetic mean' and 'Convex linear combinations'.

Theorem 6 If $f_j(z)=z^p-\sum_{n_j=k}^{\infty}|a_{n_j+p}|z^{n+p}\in P_k(p,A,B)$, then $h(z)=z^p-\sum_{n=k}^{\infty}|b_{n+p}|z^{n+p}$ also belongs to $P_k(p, A, B)$, wkere $b_{n+p}=\frac{1}{m}\sum_{j=1}a_{n_j+p}$.

Proof Since $f_j\in P_k(p, A, B)$ it follows from theorem 2 that

$$\sum_{n_j=k}^{\infty}[(1-B)n_j+(A-B)p]|a_{n_j+p}|-(A-B)p\leq 0,\quad j=1,2,\cdots m.$$

and

$$b_{n+p}=\frac{1}{m}\sum_{j=1}^{m}a_{n_j+p}.$$

Therefore we have $\sum_{n=k}^{\infty}[(1-B)n+(A-B)p]|b_{n+p}|$

$$=\sum_{n=k}^{\infty}[(1-B)n+(A-B)p]\,|\frac{1}{m}\sum_{j=1}^{m}a_{n_j+p}$$

$$\leq\sum_{n=k}^{\infty}[(1-B)n+(A-B)p]\frac{1}{m}\sum_{j=1}^{m}|a_{n_j+p}|.$$

$$\leq(A-B)p\ (\text{Since } |b_{n+p}|<\frac{1}{m}\sum_{j=1}^{m}|a_{n_j+p}|)$$

Hence from theorem 2 $h(z)$ belongs to $P_k(p, A, B)$.

Theorem 7 Let $f_p(z)=z^p$ and $f_{n+p}(z)=z^p-\frac{(A-B)p\,z^{n+p}}{[(1-B)n+(A-B)p]}$ Then $f\in P_k(p, A, B)$ if and only if it can be expressed in the form

$$f(z)=\lambda_1 f_p(z)+\sum_{n=k}^{\infty}\lambda_n f_{n+p}(z)$$

where $\lambda_n\geq 0$ and $\lambda_1+\sum_{n=k}^{\infty}\lambda_n=1$.

Proof Let us suppose that

$$f(z)=\lambda_1 f_p(z)+\sum_{n=k}^{\infty}\lambda_n f_{n+p}(z)$$

$$=(1-\sum_{n=k}^{\infty}\lambda_n)z^p+\sum_{n=k}^{\infty}\lambda_n\left[z^p-\frac{(A-B)p\,z^{n+p}}{(1-B)n+(A-B)p}\right]$$

$$= z^p - \frac{\sum_{n=k}^{\infty}\lambda_n(A-B)p z^{n+p}}{[(1-B)n+(A-B)p]} .$$

Then from theorem 2 we have

$$\sum_{n=k}^{\infty}[(1-B)n+(A-B)p]\{\frac{(A-B)p\lambda_n}{(1-B)n+(A-B)p}\}\leq(A-B)p\sum_{n=k}^{\infty}\lambda_n\leq(A-B)p$$

since $\sum_{n=k}^{\infty}\lambda_n = 1-\lambda_1 \leq 1$.

Hence $f\in P_k(p, A, B)$.

Conversely, suppose $f\in P_k(p, A, B)$. It follows from theorem 2 that $|a_{n+p}| \leq \frac{(A-B)p}{(1-B)n+(A-B)p}$. Setting

$$\lambda_n = [\frac{(1-B)n+(A-B)p}{(A-B)p}] \qquad (1)$$

where $n=k, k+1$ and $\lambda_1 = 1-\sum_{n=k}^{\infty}\lambda_n$ we have

$$f(z) = z^p - \sum_{n=k}^{\infty}|a_{n+p}|z^{n+p} = z^p - \sum_{n=k}^{\infty}\lambda_n z^{n+p} + \sum_{n=k}^{\infty}\lambda_n z^{n+p} - \frac{\sum_{n=k}^{\infty}\lambda_n(A-B)p z^{n+p}}{[(1-B)n+(A-B)p]}$$

$$= z^p(1-\sum_{n=k}^{\infty}\lambda_n) + \sum_{n=k}^{\infty}\lambda_n[z^p \frac{(A-B)p z^{n+p}}{[(1-B)n+(B-A)p]}]$$

$$= \lambda_1 f_p + \sum_{n=k}^{\infty}\lambda_n f_{n+p}(z).$$

Hence the theorem.

References

[1] R. M. Goel and N. S. Sohi (1981), Indian J.Pure Appl., Math 12, 844—853

[2] S. M. Sarangi and B. A. Uralagaddi (1978), Rendiconti Accademia Nazionaledei Lincei, LXV 38—42.

[3] S. L. Shukla and Dashrath (1982), Soochow. J.Math. 8, 179—188.

[4] Herb. Silverman (1975), Proc. Amer. Math. Soc. 51, 109—116.

[5] Vinod Kumar (1984), Journal of Mathematical Research and Exposition, Volume 4, No.1, 27—33.