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# The Radius of Univalalence of Convex Combinations of Certain Analytic Functions*

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For $a>0$, the convex combinations $f(z)=(1-a)F(z)+azF'(z)$, where $F$ belongs to different subclasses of univalent functions with fixed second coefficients are considered and the radius for which $f$ is in the same class is determined.

## 1. Introduction

The writting of this paper has been motivated by recent papers of Al-Amiri [1] and Noor, Aloboudi and Aldihan [5].

Let $S$ denote the class of all analytic and univalent functions $F$ in the unit disk $E=\{z: |z|<1\}$ which are normalized by the conditions $F(0)=0$ and $F'(0)=1$. Let $S^*$ and $K$ denote subclasses of $S$ consisting of starlike and convex functions respectively and $\overline{S}$ the subclass of $S$ consisting of functions $F$ with the property that $\mathrm{Re}(F'(z))>0$ for $z\in E$. Livingston [2] has studied the mapping propertiies of $f$: $f(z)=\frac{1}{2}[zF(z)]'$ where $F$ belongs to $S^*$, $K$ or $\overline{S}$. Al-Amiri [1] has improved most of Livingston's results by involving the second coefficient in the power expansion of $F(z)$.

In this paper we shall study the mapping properties of $f$:

$$f(z)=(1-a)F(z)+azF'(z),\quad a>0$$

when $F$ belongs to $S^*$, $K$ or $\overline{S}$, with fixed second coefficient. By taking $a=\frac{1}{2}$ we obtain the earlier results of Al-Amiri [1]. Also some of the results obtained here will generalize the earlier results due to Livingston [2] and Noor, Aloboudi and Aldihan [5].

## 2. Preliminary results

**Lemma 1** Let $\varphi(z)=b_1+b_2z+\cdots$ be an analytic map of the unit disk $E$ into itself. Then

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$$|\varphi(z)| \leq \frac{(|b_1|+|z|)}{(1 \times |b_1||z|)} \quad \text{for all } z \text{ in } E.$$

This lemma may be found in [4], p. 167.

**Lemma 2** Let $w(z) = z\varphi(z) = b_1 z + b_2 z^2 + \cdots$ be an analytic map of the unit disk $E$ into itself. Then

$$|w'(re^{i\theta})| \leqslant \frac{(r+|\varphi(re^{i\theta})|)(1-r|\varphi(re^{i\theta})|)}{(1-r^2)}.$$

This lemma is due to Al-Amiri [1].

**Lemma 3** If $P(z) = 1 + bz + \sum_{n=2}^{n} a_n z^n$ is regular and has $\mathrm{Re}(P(z)) > 0$ for $|z| < 1$, then

$$\mathrm{Re}(P(z)) \geq \frac{1-|z|^2}{1+b|z|+|z|^2} \quad \text{where } b \geq 0.$$

Further this result is sharp for each value of $b$. $0 \leq b \leq 2$ by considering the functions $P_b(z) = \dfrac{1-z^2}{1-bz+z^2}$.

This lemma is due to Tepper [6].

**Theorem A** If $P(z) = 1 + bz + \sum_{n=2}^{\infty} a_n z^n$ is regular and has $\mathrm{Re}(P(z)) > 0$ for $|z| < 1$, then

$$|P'(z)| \leq \frac{\mathrm{Re}(P(z))}{1-|z|^2}\left\{\frac{b|z|^2+4|z|+b}{|z|^2+b|z|+1}\right\}, \quad \text{where } b \geq 0.$$

This theorem is due to McCarty [3].

## 3. Main results

Without loss of generality, we will replace the second coefficient $a_2$ in the power expansion of $F(z)$ by $|a_2|$ and prove the following theorem:

**Theorem 1** Let $F(z) = z + 2pz^2 + \cdots$ be a member of the class $S^*$. Let $f(z)$ be defined by $f(z) = (1-a)F(z) + azF'(z)$, $a > 0$. Then $p \leq 1$ and $f(z)$ is starlike for $|z| < r_0$, where $r_0$ is the least positive root satisfying $(1-2a)r^4 + 2pr^3 + 6ar^2 - 2p(1-2a)r - 1 = 0$. This result is sharp.

**Proof** Since $F \in S^*$, it is well known that $2p \leq 2$ or $p \leq 1$. We can write

$$f(z) = (1-a)F(z) + azF'(z) \tag{1}$$

as

$$f(z) = az^{2-\frac{1}{a}}\left(z^{\frac{1}{a}-1}F(z)\right)'$$

and from this it follows that

$$F(z) = \frac{1}{a} z^{1-\frac{1}{a}} \int_0^z z^{\frac{1}{a}-2} f(z)\,dz. \tag{2}$$

A computation of (2) yields

$$z\frac{F'(z)}{F(z)}=\left(1-\frac{1}{a}\right)+\frac{z^{\frac{1}{a}-1}f(z)}{\int_0^z z^{\frac{1}{a}-2}f(z)\,dz} \tag{3}$$

Since $F\in S^*$, the equation (3) may be written as

$$h(z)\int_0^z z^{\frac{1}{a}-2}f(z)\,dz=\left(1-\frac{1}{a}\right)\int_0^z z^{\frac{1}{a}-2}f(z)\,dz+z^{\frac{1}{a}-1}f(z)$$

where $z\frac{F'(z)}{F(z)}=h(z)$.

Differentiating again we obtain

$$z\frac{f'(z)}{f(z)}=h(z)+h'(z)\frac{\int_0^z z^{\frac{1}{a}-2}f(z)\,dz}{z^{\frac{1}{a}-2}f(z)}. \tag{4}$$

From theorem A

$$|h'(z)|\leq\frac{\mathrm{Re}h(z)}{(1-r^2)}\left\{\frac{2pr^2+4r+2p}{r^2+2pr+1}\right\},\quad |z|=r.$$

Then from (4) we have

$$\mathrm{Re}\left(z\frac{f'(z)}{f(z)}\right)\geq\mathrm{Re}h(z)\left\{1-\frac{2(pr^2+2r+p)}{(1-r^2)(r^2+2pr+1)}\left|\frac{\int_0^z z^{\frac{1}{a}-2}f(z)\,dz}{z^{\frac{1}{a}-2}f(z)}\right|\right\} \tag{5}$$

From (1) and (2), we obtain

$$\frac{z^{\frac{1}{a}-1}f(z)}{\int_0^z z^{\frac{1}{a}-2}f(z)\,dz}=\frac{az(z^{\frac{1}{a}-1}F(z))'}{a(z^{\frac{1}{a}-1}F(z))}=z\frac{F'(z)}{F(z)}+\frac{1}{a}-1=h(z)+\frac{1}{a}-1.$$

Since $F\in S^*$, by lemma 3. $\mathrm{Re}h(z)\geq\frac{1-r^2}{1+2pr+r^2}$. Thus

$$\left|\frac{z^{\frac{1}{a}-1}f(z)}{\int_0^z z^{\frac{1}{a}-2}f(z)\,dz}\right|\geq\mathrm{Re}\left(\left(\frac{1}{a}-1\right)+h(z)\right)\geq\frac{1}{a}-1+\frac{1-r^2}{1+2pr+r^2}$$

$$=\frac{\left(\frac{1}{a}-1\right)(1+2pr+r^2)+1-r^2}{(1+2pr+r^2)}. \tag{6}$$

Inequality (5) in connection with (6) gives

$$\mathrm{Re}\left(z\frac{f'(z)}{f(z)}\right)\geq\mathrm{Re}h(z)\left[1-\frac{2r(pr^2+2r+p)}{(1-r^2)\left\{\left(\frac{1}{a}-1\right)(1+2pr^2)+1-r^2\right\}}\right]$$

$$=\mathrm{Re}h(z)\left[\frac{(1-r^2)\{(1/a-1)(1+2pr+r^2)+1-r^2\}-2r(pr^2+2r+p)}{(1-r^2)\{(1/a-1)(1+2pr+r^2)+1-r^2\}}\right] \tag{7}$$

The right hand side of ( 7 ) is positive for $r<r_0$, where $r_0$ is the least positive root of

$$(1-2a)r^4+2pr^3+6ar^2-2p(1-2a)r-1<0, \tag{8}$$

which gives the repuired root.

For $P=1$, $(1-2a)r^4+2r^3+6ar^2-2(1-2a)r-1<0$ and we get the radius of starlikeness obtained by Noor, Aloboudi and Aldihan [ 5 ]. Putting $a=\frac{1}{2}$ and $p=1$ in ( 8 ) we get $2r^3+3r^2-1<0$ which implies $r_0=\frac{1}{2}$([ 2 ]).

To prove the result is sharp, consider the function

$$F(z)=\frac{z}{1-2pz+z^2}=z+2pz^2+\cdots.$$

It is easy to see that $F\in S^*$. Further we have

$$f(z)=(1-a)F(z)+azF'(z)=\frac{z(1-2p(1-a)z+(1-2a)z^2)}{(1-2pz+z^2)^2}.$$

A computation yields

$$z\frac{f'(z)}{f(z)}=\frac{1-2p(1-2a)z-6az^2+2pz^3-(1-2a)z^4}{(1-2p(1-a)z+(1-2a)z^2)(1-2pz+z^2)}$$

and it follows that $z\frac{f'(z)}{f(z)}=0$ for $z=-r_0$. This shows that $f(z)$ is not starlike in any circle for $|z|<r$ if $r>r_0$.

**Theorem 2** Let $F(z)=z+pz^2+\cdots$ be a member of the class $K$ and $f(z)=(1-a)F(z)+azF'(z)$ for $a>0$. Then $p\leq 1$ and $f(z)$ is convex for $|z|<r_0$, where $r_0$ is the same as in theorem 1. This result is sharp.

**Proof** Since $F(z)$ is convex, it is well known that $p\leq 1$. To show that $f(z)$ is convex for $|z|<r_0$, we observe that

$$zf'(z)=az^{2-\frac{1}{a}}\left(z^{\frac{1}{a}-1}(zF'(z))\right)'. \tag{9}$$

Since $zF'(z)$ is starlike in $E$, theorem 1 and ( 9 ) shows that $zf'(z$ is starlike for $|z|<r_0$, where $r_0$ is the same as in theorem 1. Thus it follows that $f(z)$ is convex in $|z|<r_0$.

The result is sharp for the function $F(z)$ for which

$$zF'(z)=\frac{z}{1-2pz+z^2}, \text{where } p\leq 1.$$

It is easy to verify that $F\in K$. Using ( 9 ) and a similar computation to that performed in the proof of theorem 1, one gets

$$z\frac{f''(z)}{f'(z)}+1=\frac{1-2p(1-2a)z-6az^2+2pz^3-(1-2a)z^4}{(1-2pz+z^2)(1-2p(1-a)z+(1-2a)z^2)}.$$

Hence $z\frac{f''(z)}{f'(z)}+1=0$ for $z=-r_0$. This shows that $f(z)$ is not convex in any circle for $|z|<r$ if $r>r_0$.

**Theorem 3** Let $F(z)=z+pz^2+\cdots$ be a member of the class $\overline{S}$ and $f(z)=(1-a)F(z)+azF'(z)$ for $a>0$. Then $p\leq 1$ and $\mathrm{Re}(f'(z))>0$ for $|z|<r_0$, where $r_0$ is the smallest positive root satisfying

$$r^4+2p(1+a)r^3+4ar^2-2p(1-a)r-1=0.$$

This result is sharp.

**Proof** Let $F'(z)=g(z)=1+2pz+\cdots$ where $\mathrm{Re}(g(z))>0$ for $|z|<1$. Then it is well know that $2p\leq 2$ for $p\leq 1$ (see [4], p. 170). Also we have

$$f'(z)=F'(z)+azF''(z)=g(z)+azg'(z).$$

To show that $\mathrm{Re}(f'(z))>0$ for $|z|<r_0$, it suffices to show that

$$\mathrm{Re}(g(z))>a|zg'(z)| \text{ for } |z|<r_0. \tag{10}$$

Since $\mathrm{Re}(g(z))>0$, there exists an analytic function $w(z)$ in $E$ such that $|w(z)|<1$ and

$$\frac{g(z)-1}{g(z)+1}=w(z)=z\varphi(z)=pz+\cdots$$

It follows that

$$g(z)=\frac{1+w(z)}{1-w(z)}. \tag{11}$$

Thus

$$g'(z)=\frac{2w'(z)}{(1-w(z))^2}$$

Using the lemma 2, we have

$$|g'(z)|\leq\frac{2(r+|\varphi(z)|)(1-|w(z)|)}{(1-r^2)|1-w(z)|^2} \tag{12}$$

where $\varphi(z)=p+\cdots$ and $|z|=r$. Also from (11) we have

$$\mathrm{Re}(g(z))=\frac{1-|w(z)|^2}{|1-w(z)|^2}. \tag{13}$$

In view of (12) and (13) inequality (10) is satisfied if

$$\frac{1-|w(z)|^2}{|1-w(z)|^2}\geq 2ar\frac{(r+|\varphi(z)|)(1-|w(z)|)}{(1-r^2)|1-w(z)|^2} \text{ for } |z|<r_0. \tag{14}$$

Simplification of (14) yields

$$\frac{1-r^2}{2ar}\geq\frac{r+|\varphi(z)|}{1+r|\varphi(z)|}.$$

Using lemma 1 in the above inequality, we get

$$\frac{1-r^2}{2ar}\geq\frac{p+2r+pr^2}{1+2rp+r^2}.$$

This reduces to $r^4+2p(1+a)r^3+4ar^2-2p(1-a)r-1<0$, which gives the required root $r_0$.

The result is sharp for $F(z)$ for which

$$F'(z)=\frac{1-z^2}{1-2pz+z^2}=1+2pz+\cdots.$$

It is easy to see that $F\in\overline{S}$. Further

$$\begin{aligned} f'(z) &= F'(z)+azF''(z) \\ &= \frac{1-2p(1-a)z-4az^2+2p(1+a)z^3-z^4}{(1-2pz+z^2)^2}. \end{aligned}$$

Then we have $f'(z)=0$ for $z=-r_0$. This shows that $\mathrm{Re}(f'(z))\not>0$ in any circle $|z|<r$ if $r>r_0$.

## References


[1] H. S. Al-Amiri, Colloq. Math. 28(1973), 133-139.

[2] A. E. Livingston, Proc. Amer. Math. Soc. 17(1966), 352-357.

[3] C. P. McCarty, Proc. Amer. Math. Soc. 35(1) (1972), 211-216.

[4] Z. Nehari, Conformal mapping, New-York 1952.

[5] K. I. Noor, F. M. Aloboudi and N. Aldihan, Internat. J. Math. Math. Sci. Vol.6 No 2(1983), 335-340.

[6] D. E. Tepper, Tran. Amer. Math. Soc. 150(1970), 519-528.