Journal of Mathematical Research and Exposition
Vol. 10 No. 3 Aug. 1990

On L^2-Boundedness of Singular Integral with Oscillating Kernel*

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Abstract In this paper, we consider L^2-boundedness of singular integrals with oscillating kernels and corresponding $T(1)$-Theorem, and generalize or improve some results of Hu, Phong and Stein.

I. Introduction

Let $K\in D'(R^n\times R^n)\cap C(R^n\times R^n-\{(x,x):x\in R^n\})$, $B: R^n\times R^n\to R^1$ s.t. $(x,y)\to xBy^T$, where B is a symmetric (real) matrix, define a singular integral operator from $D(R^n)$ to its dual continuously as follows

$$S_B(f)(x)=\int_{R^n}K(x,y)e^{iB(x,y)}f(y)\mathrm{d}y \tag{1}$$

where $f\in D(R^n)'$. In this paper, we consider L^2-boundedness of S_B.

Notations are as follows.

$K\in D^\infty_{a,k,\mu}$ $(a,\mu\geqslant 0,\ k=1,2,3,\cdots)$ implies

$$|\nabla_x^l K(x,y)|+|\nabla_x^l K(y,x)|\leqslant C_{k,l}(1+|x-y|)^{-\mu-l}\quad (0\leqslant l\leqslant k-1,\ |x-y|\geqslant 1)$$

$$|\nabla_x^{k-1}K(x+h,y)-\nabla_x^{k-1}K(x,y)|+|\nabla_x^{k-1}K(y,x+h)-\nabla_x^{k-1}K(y,x)| \tag{2}$$

$$\leqslant C_{K,k}(1+x-y|)^{-\mu-(k-1)}\cdot|h|^a\qquad (|h|<\frac{1}{2}|x-y|,\ |x-y| \text{ and } |x+h-y|\geqslant 1)$$

$K\in D^0_\omega$ implies

$$|K(x,y)|\leqslant C_k\cdot|x-y|^{-n}\qquad (|x-y|\leqslant 2)$$

$$|K(x+h,y)-K(x,y)|+|K(y,x+h)-K(y,x)|\leqslant C_k\cdot|x-y|^{-n}\omega(\frac{|h|}{|x-y|}) \tag{3}$$

$$(\forall\,|h|\leqslant\frac{1}{2}|x-y|\leqslant 1)$$

$K\in D_{a,k,\mu}$ or D_ω implies (3) or (2) without restrictions $|x-y|$ and $|x+h-y|\geqslant 1$ or $\leqslant 2$". $\omega\in D^*$ implies that ω is increasing in $[0,+\infty)$ and

$$\int_0^1\frac{\omega(\delta)}{\delta}\ln\frac{1}{\delta}\mathrm{d}\delta<\infty.$$

$T\in WBP^0$ implies that T is a continuous linear operator from $D(R^n)'$ to $D'(R^n)'$,

* Received Sept. 16, 1988.

and

$$|\langle Th_t^x, g_t^y\rangle| \leqslant C_{T,h,g} t^{-n} \qquad (0<t\leqslant 1,\ h,g\in D(R^n)')$$

where $h_t^x(y)=t^{-n}h(\frac{y-x}{t})$, g_t^y is similar. Without the restriction "$t\leqslant 1$", we write $T\in WBP$. Finally, fix $\psi_0\in D(R^n)$ such that $\psi_0|_{|x|<\frac{1}{2}}\equiv 1$, $\psi_0|_{|x|>1}\equiv 0$, then define

$$T_B(f)(x)=\begin{cases} S_B(f)(x) & \text{when rank } (B)=0 \\ \int_{R^n}K(x,y)\psi_0(x-y)f(y)\mathrm{d}y & \text{when rank}(B)>0 \end{cases} \tag{4}$$

When rank$(B)=0$, David-Journe[1] got following famous $T(I)$-Theorem.

Theorem A Let $K\in D_\omega, \omega(\delta)=\delta^a$ $(0<a<1)$, then S_B is L^2-bounded iff $T_B(I)$ and $T_B^*(I)\in BMO$ and $T_B\in WBP$.

In [4,5], Yabuta and author improved Theorem A.

When rank $(B)=n$, Hu Yu[2] got a corresponding $T(I)$-Theorem.

Theorem B Suppose rank$(B)=n$, $K\in D_\omega$ or $K\in D_\omega^0$ but K has bounded derivitives of sufficiently higher orders, $\omega(\delta)=\delta$, then S_B is L^2-bounded iff $T(I)$ and $T_B^*(I)\in BMO^0$ and $T_B\in WBP^0$.

For general B, Phong and Stein proved

Theorem C If $K(x,y)=\widetilde{K}(x-y)$, $\widetilde{K}\in C^N(R^n)$, and $|\nabla^l\widetilde{K}(x)|\leqslant C_K$, $(1+|x|)^{-\mu-l}$ $(0\leqslant l\leqslant N)$, then S_B is L^2-bounded if $\mu>n$-rank (B) and N is sufficiently large.

In this paper, we prove

Theorem I If $k=\text{rank}(B)>0$, $K\in D_{a,k,\mu}$, $0<a\leqslant 1$, $\mu>n-k+(1-a)$, then S_B is L^2-bounded.

Theorem II If $k=\text{rank}(B)>0$, $K\in D_{a,k,\mu}^{\infty}\cap D_\omega^0$, $\omega\in D^*$, then, S_B is L^2-bounded iff $T_B(I)$ and $T_B^*(I)\in BMO^0$ and $T_B\in WBP^0$.

Theorem I improves Theorem C, Theorem II improves and generalizes Theorem B.

II. Proofs

Theorem II is a corollary of Theorem I and [4; Theorem 2], so, it is enough to prove Theorem I. Take a nonnegative, radial C^∞-function ψ such that supp $\psi \subset\{\frac{1}{4}\leqslant|x|\leqslant 1\}$, $\sum_{-\infty}^{\infty}\psi_j(x)=1$ $(\forall x\neq 0)$ where $\psi_j(x)=\psi(2^jx)$. Then, set $K_0(x,y)=K(x,y)\sum_{-\infty}^{0}\psi_j(x-y)$, $K_j(x,y)=K(x,y)\psi_j(x-y)$ $(j>0)$, and

$$T_j(f)(x)=\int_{R^n}e^{iB(x,y)}K_j(x,y)f(y)\mathrm{d}y.$$

Because $\|T_j\|_{2,2}=\|T_j^*T_j\|_{2,2}^{1/2}$, and the kernel of $T_j^*T_j$ is

$$L_j(x,y)=\int_{R^n}e^{iB(x-y,u)}K_j(u,y)\overline{K}_j(u,x)\mathrm{d}u,$$

so, it is enough for Theorem I to prove

$$\sum_{j=0}^{\infty}\sup_{y}\left(\|L_j(\cdot,y)\|_1+\|L_j(y,\cdot)\|_1\right)^{1/2}<+\infty. \tag{5}$$

It is not hard to see that

$$|L_j(x,y)|\leqslant C_{K,\psi}\cdot 2^{-(2\mu-n)j}\chi_{B(0,2^{j+1})}(x-y). \tag{6}$$

Now, let P denote the orthogonal projection operator from R^n to $B(R^n)=\{Bx: x\in R^n\}$, for $\mathscr{L}_u=i(a,\nabla_u)/B(a,x-y)$ where $a\in R^n$, there holds $\mathscr{L}_u^N e^{-iB(u,x-y)}=e^{-iB(u,x-y)}$ take $a=B_1(P(x-y)/|P(x-y)|)$ where B_1 is a matrix such that $BB_1=P$ and $B_1P=B_1$, then $B(a,x-y)=|P(x-y)|$ and

$$L_j(x,y)=|P(x-y)|^{-k+1}(-i)^{k-1}\int e^{-iB(x-y,u)}(a,\nabla_u)^{k-1}(K_j(u,y)\overline{K}_j(u,x))\,du$$
$$\triangleq(-i)^{k-1}|P(x-y)|^{-k+1}\sum_{l+m=k-1}N_{j,l,m}(x,y)$$

where

$$N_{j,l,m}=\int(a,\nabla_u)^l K_j(u,y)(a,\nabla_u)^m\overline{K}_j(u,x)e^{-iB(x-y,u)}du$$
$$=e^{-iB(x-y,y)}\int_{\Sigma_{n-1}}n_{j,l,m,u'}(x,y)\,d\sigma(u')$$

and

$$n_{j,l,m,u'}(x,y)=\int_0^{2^j}e^{-iB(x-y,u')r}((a,\nabla_u)^lK_j)(y+ru',y)\cdot$$
$$((a,\nabla_u)^m\overline{K}_j)(y+ru',x)r^{n-1}dr$$
$$=2^{nj}\int_0^1 e^{-iB(x-y,u')2^jr}((a,\nabla_u)^lK_j)(y+2^ju'r,y)\cdot$$
$$((a,\nabla_u)^m\overline{K}_j)(y+2^ju'r,x)r^{n-1}dr\,\chi_{B(0,2^{j+1})}(x-y).$$

Because $K\in D^{\infty}_{a,k,\mu}$, and $\operatorname{supp}K_j\subset\{2^{j-2}\leqslant|x-y|\leqslant 2^j\}$, it is not hard to see (Remark 2)

$$\|(a,\nabla_u)^lK_j(\cdot,x)\|_{\wedge a}=\sup_{u\in R^n,h\neq 0}|h|^{-a}|(a,\nabla_u)^l\cdot$$
$$K_j(u+h,x)-(a,\nabla_u)^lK_j(u,x)|\leqslant C_{k,\psi,l}2^{-(\mu+l)j} \tag{7'}$$

$$\|(a,\nabla_u)^lK_j(\cdot,x)\|_\infty\leqslant C_{K,\psi,l}2^{-(\mu+l)j} \tag{7''}$$

$$\begin{cases}\operatorname{supp}(a,\nabla_u)^lK_j(u+\cdot,\omega)\subset B(0,2^j),\\ \operatorname{supp}(a,\nabla_u)K_j(u+\cdot,x)\subset B(0,2^{j+2})\ \text{for}\ u-x\in B(0,2^{j+1}).\end{cases} \tag{7'''}$$

And, when $\operatorname{supp}f\subset[0,2^{j+2}]$, it is also not hard to see (Remark 3)

$$\left|\int_0^1 f(2^jr)e^{-i\lambda 2^jr}r^{n-1}dr\right|\leqslant C_n(\|f\|_\infty/2^j\lambda+\|f\|_{\wedge a}/\lambda^a). \tag{8}$$

By (7′)—(7‴) and (8) and the fact that $\|fg\|_{\wedge a}\leqslant\|f\|_\infty\cdot\|g\|_{\wedge a}+\|g\|_\infty\|f\|_{\wedge a}$, we get

$$|n_{j,l,m,u'}(x-y)|\leqslant 2^{nj}C_{K,k,\psi,n}\Big(\frac{2^{-(2\mu+l+m)j}}{2^j|B(x-y,u')|}$$
$$+\frac{2^{-(2\mu+l+m)j}}{2^{aj}|B(x-y,u')|^a}\Big)\chi_{B(0,2^{j+1})}(x-y),$$

so, (6) gives (for $0<\varepsilon<1$)

$$|N_{j,l,m}(x,y)|\leqslant C_{K,k,\psi,n}2^{-(2\mu+l+m-n)\varepsilon j}\cdot 2^{-(2\mu-n)(1-\varepsilon)j}\cdot$$
$$\cdot\int_{\Sigma_{n-1}}\left(\frac{2^{-\varepsilon j}}{|B(x-y,u')|^{a}}+\frac{2^{-a\varepsilon j}}{|B(x-y,u')|^{a\varepsilon}}\right)d\sigma(u')\chi_{B(0,2^{j+1})}(x-y)$$
$$\leqslant C_{K,\psi,k,n,\varepsilon,a}\cdot 2^{-(2\mu+l+m-(2-\varepsilon)n)}(2^{-\varepsilon j}/|x-y|^{\varepsilon}$$
$$+2^{-a\varepsilon j}/|x-y|^{a\varepsilon})\chi_{B(0,2^{j+1})}(x-y).$$

Thus, for $\varepsilon\in(0,1)$, there holds

$$|L_j(x,y)|\leqslant C_{K,\psi,k,n,\varepsilon,a}\cdot 2^{-(2\mu+k-1-(2-\varepsilon)n)}\cdot$$
$$(2^{-\varepsilon j}/|(I-P)(x-y)|^{\varepsilon}+2^{-a\varepsilon j}/|(I-P)(x-y)|^{a\varepsilon})\cdot$$
$$|P(x-y)|^{-k+1}\chi_{B(0,2^{j+1})}(x-y)$$

and

$$\int(|L_j(x,y)|+|L_j(y,x)|)dx$$
$$\leqslant C_{K,\psi,k,n,\varepsilon,a}\cdot 2^{-(2\mu+k-1-(2-\varepsilon)n)}\Big(2^{-\varepsilon j}\int_{\substack{\tilde{x}\in(I-P)R^n\\|\tilde{x}|<2^{j+1}}}\frac{d\tilde{x}}{|x|^{\varepsilon}}\cdot$$
$$\cdot\int_{\tilde{\tilde{x}}\in P(R^n),\,|\tilde{\tilde{x}}|<2^{j+1}}\frac{d\tilde{\tilde{x}}}{|\tilde{\tilde{x}}|^{k-1}}+2^{-a\varepsilon j}\int_{\substack{\tilde{x}\in(I-P)R^n\\|\tilde{x}|<2^{j+1}}}\frac{d\tilde{x}}{|\tilde{x}|^{\varepsilon a}}\cdot\int_{\substack{\tilde{\tilde{x}}\in P(R^n)\\|\tilde{\tilde{x}}|<2^{j+1}}}\frac{d\tilde{\tilde{x}}}{|\tilde{\tilde{x}}|^{k-1}}$$
$$\leqslant C\cdot 2^{-(2\mu+k-1-(2-\varepsilon)n)}\cdot(2^{-\varepsilon j}2^{((n-k)-\varepsilon)-\varepsilon)j}\cdot 2^{j}$$
$$+2^{-a\varepsilon j}2^{((n-k)-\varepsilon a)j}2^{j})$$
$$\leqslant C\cdot 2^{-2(\mu+k-1-(3-\varepsilon)n/2+a\varepsilon)j}.$$

Because $\mu>n-k+(1-a)$, $\mu+k-1-(3-\varepsilon)n/2+a\varepsilon>0$ when ε is sufficiently close to 1, thus (5) holds. Theorem I is proved.

Remark 1 From the above proof, we see that if (2) is replaced by

$$|\nabla_x^{k-1}K(x+h,y)-\nabla_x^{k-1}K(x,y)|+|\nabla_x^{k-1}K(y,x+h)-\nabla_x^{k-1}K(y,x)|$$
$$\leqslant C_{k,K}(1+|x-y|)^{-\mu-(k-1)-\beta}|h|^{a}\quad\text{for}\quad |h|\leqslant|x-y|/2,$$

then, in Theorem I, "$\mu>n-k+(1-a)$" can be replaced by "$\mu>n-k+\max(0,1-a-\beta/2)$".

Corollary If $K(x,y)=\widetilde{K}(x-y)$, $\widetilde{K}\in C^{n-k}(R^n)$, $k=\operatorname{rank}(B)>0$ and

$$|\nabla^{l}\widetilde{K}(x)|\leqslant C_l(1+|x|)^{-\mu-l}\qquad(\forall\, 0\leqslant l\leqslant k-1)$$
$$|\nabla^{k}\widetilde{K}(x)|\leqslant C_l(1+|x|)^{-\mu-(k-1)}$$

then, S_B is L^2-bounded when $\mu>n-k$.

Remark 2 $(7'')$ and $(7''')$ are obvious. Now, we prove $(7')$, for example, for $k=1$ and $l=0$. Because $\operatorname{supp}K_j\subset\{2^{j-2}\leqslant|x-y|\leqslant 2^{j}\}$, we can assume that at least one of $y+h-x$ and $y-x$, for example $y+h-x$, belongs to $B(0,2^{j})-B(0,2^{j-2})$. Now, if $x-y\in B(0,2^{j+1})-B(0,2^{j-3})$, then $|h|\leqslant 2^{j+2}$, thus

$$|K_j(y+h,x)-K_j(y,x)|\leqslant\|K\|_{\infty}|\psi_j(y+h-x)-\psi_j(y-x)|$$

$$+|K(y+h,x)-K(y,x)|\|\psi\|_\infty$$
$$\leqslant C_K(\|\nabla\psi\|_\infty\cdot 2^{-\mu j}+\|\psi\|_\infty\cdot|h|^a|x-y|^{-\mu}\chi_{|h|<2^{j-4}}(h)$$
$$+\|\psi\|_\infty\|K\|_\infty\cdot\chi_{|h|>2^{j-4}}(h))$$
$$\leqslant C_{K,\psi}2^{-2\mu j}|h|^a.$$

If $x-y\in B(0,2^{j+1})-B(0,2^{j-3})$, then $|h|\geqslant 2^{j-3}$, thus

$$|K_j(y+h,x)-K_j(y,x)|=|K_j(y+h,x)|$$
$$\leqslant C_{K,\psi}2^{-\mu j}\leqslant C_{K,\psi}|h|^a\cdot 2^{-\mu j}.$$

Remark 3 Proof of (8). Because $\mathrm{supp} f\subset[0,2^{j+2}]$, $|\pi/\lambda|\leqslant 2^{j+3}$ when $f(2^jr-\pi/\lambda)\neq 0$, thus

$$\int_0^1 f(2^jr)e^{-i\lambda 2^jr}r^{n-1}\mathrm{d}r=-\int_{\frac{\pi}{2^j\lambda}}^{1+\frac{\pi}{2^j\lambda}}f(2^jr-\frac{\pi}{\lambda})(r-\frac{\pi}{2^j\lambda})^{n-1}e^{-i\lambda e^jr}\mathrm{d}r$$
$$=\frac{1}{2}\int_0^1 e^{-i\lambda 2^jr}(f(2^jr-\frac{\pi}{\lambda})(r^{n-1}-(r-\frac{\pi}{\lambda 2^j})^{n-1})$$
$$+r^{n-1}(f(2^jr)-f(2^jr-\frac{\pi}{\lambda}))\mathrm{d}r$$
$$+\frac{1}{2}(\int_0^{\frac{\pi}{2^j\lambda}}-\int_1^{1+\frac{\pi}{2^j\lambda}})f(2^jr-\frac{\pi}{\lambda})(r-\frac{\pi}{2^j\lambda})^{n-1}e^{-i\lambda 2^jr}\mathrm{d}r$$
$$=O(1)(\|f\|_\infty/2^j\lambda+\|f\|_{\wedge a}/\lambda^a+\|f\|_\infty/2^j\lambda).$$

(8) is proved.

References

〔1〕David G., Journe J.L., Ann. of Math., 120(1984), 371-397.

〔2〕Hu Yu, Function spaces and boundedness of operators with oscillatory kernels, Ph. D. Thesis, Beijing University, 1986.

〔3〕Phong D. H., Stein E. M., Acta Math., 157 (1986), 99—157.

〔4〕Xu Han, Some remarks on T(1)-Theorems, to appear in J. Northeastern Math.

〔5〕Yabuta K., Studia Math., 82(1985), 17—31.

关于振动核奇异积分的L^2-有界性

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在本文中，我们研究了形如$S_B(f)(x)=\int_{R^n}K(x,y)e^{iB(x,y)}f(y)\mathrm{d}y$的振动核奇异积分的$L^2$-有界性及相应的$T(1)$—定理，其中$B$非退化。对于相当广的一类核函数$K$，$S_B$的奇异性只取决于$K$在“0”点附近的奇异性；此外，为了建立$T(1)$—定理，我们把核函数的光滑性降到了一种近似于$D_{ini}$-条件的积分条件.