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Variational Methods In Random Analysis*

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Abstract In this paper, we consider the random variational inequality (its definition in the following), the methods we used belong to functional analysis.

1. Preliminaries

Thoughout this paper, (Ω, Σ) denotes a measurable space. E is a tolological vector space. $\Phi: \Omega \to 2^E$ is called measurable if for any open subset B of X, $\Phi^{-1}(B) := \{\omega \in \Omega,\ F(\omega) \cap B \neq \emptyset\} \in \Sigma$. Notice that $\Phi: \Omega \to 2^E$, if $\forall \omega \in \Omega$, $\Phi(\omega) \in K(X)$, then Φ is measurable if and only if $\Phi^{-1}(C) \in \Sigma$, for any closet subset C of X, in which 2^E is the family of all subset of X, $CD(X)$ all nonempty closed subset of X, $CB(X)$ all nonempty bounded closed subsets of X, $K(X)$ all nonempty compact subsets of X, respectively. A mapping $f: \Omega \times X \to 2^Y$ is called a random operator if for any $x \in X, f(\cdot, x)$ is mesurable. Random operator $T: \Omega \times X \to Y$ called continuous (compact, etc) if $\forall \omega \in \Omega$, $f(\omega, \cdot)$ is continuous (compact, etc). A measurable mapping $f: \Omega \to X$ is called a measurable selector of a measurable mapping $F: \Omega \to CD(X)$, if $\forall \varepsilon \in \Omega, f(\varepsilon) \in F(\varepsilon)$. Notice that when S is a separable closed subset of E, $F: \Omega \to 2^S$ is measurable and $F(\omega)$ is compact, then F has a measurable selector. A measurable mapping $\tilde{x}: \Omega \to X$ is called a random fixed point of a random operator $T: \Omega \times X \to 2^X$, if $\forall \omega \in \Omega$, $\tilde{x}(\omega) \in T(\omega, \tilde{x}(\omega))$. Random mapping $G: \Omega \times X \to$ $\to 2^X$ called random KKM mapping, if $\forall \omega \in \Omega$, $G(\omega, \cdot)$ is KKM. In this paper, let X be usually a convex subset

2. Random variational inequality

Theorem 1 Let E be a T.V.S. (i.e, topological vector space), $G: \Omega \times X \to 2^E$ is a random KKM mapping and $\forall \omega \in \Omega$, $x \in X$, $G(\omega, x)$ is finite closed (i.e, the interesection of $G(\omega, x)$ with any finite dimensional subspace of E is closed), then for any $\omega \in \Omega$, $\bigcap \{G(\omega, x)\} \neq \emptyset$.

Lemma 1 Let S be a compact subset of a Hausdorff locally convex topological vector space (r.L.C.T.V.S.); Then for any random continuous operator $T: \Omega \times S — S$ has a random fixed point.

Theorem 2 Let E be a T.V.S., $X \subset E$. $G: \Omega \times X \to 2^E$ is a Random KKM

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mapping, and for any $(\omega, x)\in\Omega\times X$, $G(\omega, x)$ is a closed subset, there exists a subset $X_0\subset X$ such that $\bigcap_{x\in X_0} G(\omega, x)$ is compact, $X_0\subset C$. C is a compact convex subset of E; Then $\bigcap_{x\in X}\{G(\omega, x)\}\neq\varnothing$.

Theorem 3 Let E be a T.V.S., $X\subset E$, $G:\Omega\times X\to 2^E$ is a random KKM mapping such that: (1) $\forall(\omega, x)\in\Omega\times X$, $G(\omega, x)$ is finite closed; (2) $\forall\omega\in\Omega$, $\exists x_0\in X$, $\overline{G(\omega, x_0)}$ is compact; (3) for any $D=X\cap F$(F is a finite dimensional subspace of E). $\bigcap_{y\in D} G(\omega, y)\cap D=(\bigcap_{y\in D} G(\omega, y))\cap D$. Then $\bigcap_{x\in X}\{G(\omega, x)\}\neq\varnothing$.

In the followings, we use the early results to discuss the random variational inequality, i.e, the problem (X, f, Φ):

Let E be a T.V.S., $X\subset E$ is a closed subset of E. Random mapping f: $\Omega\times X\to(-\infty, +\infty]$, and $f\not\equiv+\infty$; Random operator $\Phi:\Omega\times X\times X\to\mathbf{R}$, $\forall(\omega, x, x)\in\Omega\times X\times X$, $\Phi(\omega, x, x)\geqslant 0$; we want to find a random mapping $\tilde{x}:\Omega\to X$, such that: For any $(\omega, y)\in\Omega\times X$

$$(**)\qquad f(\omega, y)+\Phi(\omega, \tilde{x}(\omega, y)\geqslant f(\omega, \tilde{x}(\omega))$$

Then, we call the $(**)$ is a random variational inequality.

Theorem 4 Let E be a T.V.S., the f and Φ's definitions the same as problem (X, f, Φ), such that:

1. There exists a compact subset $K\subset E, x_0\in X\cap K$ s.t. For any $(\omega, x)\in\Omega\times(X\setminus K)$, $f(\omega, x)>\Phi(\omega, x, x_0)+f(\omega, x_0)$;
2. For each fixed $x\in X$, $\omega\in\Omega$, $f(\omega, y)+\Phi(\omega, x, y)$ is quasi convex on y;
3. For each fixed $y\in X$, $\omega\in\Omega$, $f(\omega, x)-\Phi(\omega, x, y)$ is uniformly continuous on y; Then, problem $(**)(x, f, \Phi)$ has a random solution. i.e., there exists a random mapping $\tilde{x}:\Omega\to X\cap K$.

By using theorem 4, we have

Corollary 1 For problem $(**)$, let X be a compact subset, and:

1. For each fixed $x\in X$, $\omega\in\Omega$, $f(\omega, y)+\Phi(\omega, x, y)$ is quasi-covex on y;
2. For any $x\in X$, $y\in X$, $f(\omega, x)-\Phi(\omega, x, y)$ is uniformly lower semi continuous on x;

Then problem (X, f, Φ) has a random solution.

Corollary 2 Let E be a reflexive semi-normed space, s.t.

1). For each $x\in X$, $\omega\in\Omega$, $f(\omega, y)+\Phi(\omega, x, y)$ is quasi convex on y;

2). For any $y\in X$, $\omega\in\Omega$, $f(\omega, x)-\Phi(\omega, x, y)$ is uniformly continuous on x;

Then the problem (x, f, Φ) has a random solution.

Definition Random operator Φ ; $\Omega\times X\times X\to\mathbf{R}$ called random monotone if for any $(\omega, x, y)\in\Omega\times X\times X$, we have $\Phi(\omega, x, y)+\Phi(\omega, y, x)\geqslant 0$.

Befor geting the results of random monotone varational inequality,we need lemma 2.

Lemma 2 Let E be a T. V. S., X be a closed subset of E. $f: \Omega\times X\to[-\infty, +\infty]$, $f(\omega,x)\not\equiv+\infty$ and continuous on x; $\Phi:\Omega\times X\times X\to\mathbf{R}$ is monotone semi-continuous on (x,y), such that for any $x\in X$, $\Phi(\omega,x,x)\geqslant 0$, and for each fixed $x\in X$, $\omega\in\Omega$, $\Phi(\omega,x,y)$ is continuous on y; If for each $\omega\in\Omega$, $y\in X$, $f(\omega,y)+\Phi(\omega,x,y)$ is convex on y. Then the following conditions are equivalent:

1). For each $\omega\in\Omega$, there exists x_ω, such that for any y:

$$f(\omega,y)+\Phi(\omega,x_\omega,y)\geqslant f(\omega,x_\omega);$$

2). For each $\omega\in\Omega$, the x_ω s.t. $f(\omega,y)-\Phi(\omega,y,x_\omega)\geqslant f(\omega,x_\omega)$, for any $y\in X$, in particular, if X is compact. In this case, if there exists a random mapping $\tilde{x}:\Omega\to X$ such that $f(\omega,y)+\Phi(\omega,\tilde{x}(\omega),y)\geqslant f(\omega,\tilde{x}(\omega))$ for any $y\in x$, $\omega\in\Omega$ if and only if $f(\omega,y)-\Phi(\omega,y,\tilde{x}(\omega))\geqslant f(\omega,\tilde{x}(\omega))$ for any $y\in X$, $\omega\in\Omega$.

Theorem 5 The definitions of X, f, Φ are the same as theorem 4, suct that:

1). There exists compact subset $K\subset X$, and $x_0\in K$, s.t. $f(\omega,x)\geqslant\Phi(\omega,x,x_0)+f(\omega,x_0)$ for any $x\in X\setminus K$, $\omega\in\Omega$;

2). For each $\underset{x\in X}{\omega\in\Omega}$, $f(\omega,y)+\Phi(\omega,x,y)$ is convex on $y\in X$;

3). For any $\omega\in\Omega$, $x\in X$, $f(\omega,x)-\Phi(\omega,x,y)$ is uniformly on x (in particular, for any $\omega\in\Omega$, $y\in X$, $f(\omega,y)$, $\Phi(\omega,x,y)$ is lower semicontinuous on $y\in X$, respectively);

4). For each $\omega\in\Omega$, $\Phi(\omega,x,y)$ is monotone semicontinuous.

Then the problem (X,f,Φ) has random solution, and the set of all random solution has the property: for any fixed $\omega\in\Omega$, the mapping image of all random solution is a compact subset of $X\cap K$. If $\Phi(\omega,x,y)$ is strictly monotone, then the problem (x,f,Φ) has a only random solution.

In fact, corollary 1 is the random minmax inequality of Fan Ky type, here we write it as the following type.

Theorem 6 Let E be a T. V. S., X is a closed convex compact of E, such that for any $\omega\in\Omega$, $\Phi(\omega,x,x)\geqslant 0$, and s.t.

1). For each $x,\omega\in X$, $f(\omega,y)+\Phi(\omega,x,y)$ is quasi-convex on y;

2). For any $\omega\in\Omega$, $x\in X$, $f(\omega,x)-\Phi(\omega,x,y)$ is uniformal lower-semicontinuous on x. Then there exists a random mapping: $\tilde{x}:\Omega\to X$ s.t.: $f(\omega,y)+\Phi(\omega,\tilde{x}(\omega),y))\geqslant f(\omega,\tilde{x}(\omega))$.

Theorem 7 Let E be a seminormed space, $X\subset E$ is a closed compact of E, $T:\Omega\times X\to E$ is a random continuous operator. Then either (1) or (2) is true:

(1) For any $\omega\in\Omega$, there exists $\tilde{x}_0(\omega)$ s.t. $T(\omega,\tilde{x}_0(\omega))=\tilde{x}_0(\omega)$;

(2) There exists $\bar{x}\in X$ and a continuous seminorm P such that: $0<P(\bar{x}-$

$T(\omega, \tilde{x})) = \min_{x \in X} P(x - T(\omega, x))$, in which: $\tilde{x}_0: \Omega \to X$ is meosurable.

Corellary 3 Let E be a seminormed space, $X \subset E$ is a closed compact subset and $T: \Omega \times X \to X$ is random continuous. Then T has a random fixed point.

Remark Cor.3 is the random fixed point of Schauder type.

Definition Let $\Phi: \Omega \times X \times Y \to \mathbf{R}$, the random mapping $\tilde{x}: \Omega \to X$, $\tilde{y}: \Omega \to Y$ are called the random saddle-point mapping of Φ if for any fixed $\omega \in \Omega$, such that: For any $(x, y) \in X \times Y$, $\Phi(\omega, x, \tilde{y}(\omega)) \leqslant \Phi(\omega, \tilde{x}(\omega), \tilde{y}(\omega)) \leqslant \Phi(\omega, \tilde{x}(\omega), y)$.

Theorem 8 Let E, F be the T.V.S., $X \subset F$, and X, Y are the compact convex subset of E, F respectively. For each fixed $\omega \in \Omega$. $\Phi(\omega, x, y)$ is continuous on (x, y). Then the (1) is true if and only if the (2) is true.

(1) For each $\omega \in \Omega$, $\min_{y \in Y} \max_{x \in X} \Phi(\omega, x, y) = \max_{x \in X} \min_{y \in Y} \Phi(\omega, x, y)$

(2) For each $\omega \in \Omega$, there exists the random saddle-point mapping $\tilde{x}$, $\tilde{y}$ such that $\Phi(\omega, x, \tilde{y}(\omega)) \leqslant \Phi(\omega, \tilde{x}(\omega), \tilde{y}(\omega)) \leqslant \Phi(\omega, \tilde{x}(\omega), \tilde{y})$.

Corollary 4 The E, F, X, Y, Φ, the same as the theorem 8, if

(1) For each $\omega \in \Omega$, $x \in X$, $\Phi(\omega, x, y)$ is continuous and concave on y.

(2) For each $\omega \in \Omega$, $y \in X$, $\Phi(\omega, x, y)$ is continuous and convex on x;

Then there exists random saddle-point mapping $\tilde{x}: X \to \mathbf{R}$; $\tilde{y}: Y \to \mathbf{R}$. such that $\Phi(\omega, \tilde{x}(\omega), y) \leqslant \Phi(\omega, \tilde{x}(\omega), \tilde{y}(\omega)) \leqslant \Phi(\omega, x, \tilde{y}(\omega))$.

3. Quasi-random variational inequality

In this section, we get some results on the quasi-random variational inequality, in this field, some authors have the results, for example [7], [2], [9], [11], we first have the random fixed point theorem of Browder type.

Theorem 9 Let E be a T.V.S., X is a compact convex of E. S: $\Omega \times X \to 2^X$ is a continuous set-valued mapping such that 1) or, 2) is true:

1) For each ω, $x \in X$, $s(\omega, x)$ is a nonempty convex subset of X, and for any $y \in X$, $S_\omega^{-1}(y) = \{x \in X, y \in s(\omega, x)\}$ is a open subset of X;

2) For each $\omega \in \Omega$, $x \in X$, $S(\omega, x)$ is a open subset of X, and for any $y \in X$, $S_\omega^{-1}(y)$ is a nonempty closed convex subset of X.

Then S has a random fixed point.

Theorem 10 Let E be a semi-normed space, X is a nonempty compact set of E; Φ: $\Omega \times X \to 2^X$ is nonempty closed convex and upper semi-continuous set-valued. Then Φ has random fixed point.

Definition The quasi-variational inequality is the following problem $(\Omega, X, X_0, g, \Phi)$.

Let E be a semi-normed linear space, X, X_0 are closed convex subset of E and $X_0 \subset X$, $g: \Omega \times X_0 \times X \to (-\infty, +\infty]$ is a random mapping and for any $z \in X_0$, $\exists x \in X$, such that $g(\omega, z, x) < +\infty$, and $\Phi: \Omega \times X_0 \times X \times X \to \mathbf{R}$ is a random mapping such that for any $z \in X_0$, $x \in X$, $\Phi(\omega, z, x, x) \geqslant 0$. We want to find a random $\tilde{x}: \Omega \to X$ such that for any $y \in X$,

$$g(\omega, \tilde{x}(\omega), y) + \Phi(\omega, \tilde{x}(\omega), \tilde{x}(\omega), y) \geqslant g(\omega, \tilde{x}(\omega), \tilde{x}(\omega)).$$

Theorem 11 If the following conditions are true, then the problem$(\Omega, X, X_0, g, \Phi)$ has a random solution.

(A) X is a compact set and for each $\omega, z \in X$, $\Phi(\omega, z, x, y)$ is monotone and continuous on (x, y). $\Phi(\omega, z, x, y)$ and $g(\omega, z, y)$ is convex on y;

(B) For each ω, $\Phi(\omega, z, x, y)$ is continuous on $(x\ y)$.

Corollary 5 Let E be a seminormed linear space, $X \subset E$ is a nonempty convex subset, $J: \Omega \times X \times X \to \mathbf{R}$ and for each $\omega \in \Omega$, the random mapping $J(\omega, x, y)$ is continuous and monotone, for any $\omega \in \Omega$, $x \in X$, $J(\omega, x, y)$ is convex on y and $J(\omega, x, x) = 0$; $\Phi: \Omega \times X \to 2^X$ is a nonempty closed continuous set-valued random mapping such that: if $x_0 \in \Phi(\omega, z_0)$, we have that for any $y \in \Phi(\omega, z_0)$ the inequality $J(\omega, x_0, y) \geqslant 0$ is true. Then we know that: there exists a random fixed point of Φ, i.e. $\tilde{x}(\omega) \in \Phi(\omega, \tilde{x}(\omega))$, and $J(\omega, \tilde{x}(\omega), y) \geqslant 0$, $\forall y \in Q(\omega, \tilde{x}(\omega))$.

In the end, we have

Corollary 6 Let E be a semi-normed linear space, X is a compact of E and its convex. $\Phi: \Omega \times X \to 2^X$ is nonempty closed continuous, $J: \Omega \times X \times X \to \mathbf{R}$ also random continuous and for any $\omega \in \Omega$, $x \in X$, $J(\omega, x, y)$ is convex on y. For any $\omega \in \Omega$, $x \in X$, $J(\omega, x, x) = 0$. Then Φ has a random fixed point, i.e., for any $\omega \in \Omega$, $x(\omega) \in \Phi(\omega, x(\omega))$.

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