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A State-Constrained Minimum-Energy Optimal Control Problem for a Steady-State PDE System*

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Abstract A minimum-energy optimal control problem with inequality-or smoothing-constraint for a steady-state system described by harmonic partial differential equation is studied. Complete and closed-form optimal solutions are obtained via a spline-based technique developed recently by the author ([1,2]). It is shown that the optimal solutions have an elegant reproducing kernel structure.

Ⅰ. Introduction

The spline approach based on the technique of reproducing kernel Hilbert space is quite successful in obtaining explicit closed-form solutions for constrained optimal control problems. In this context, the reader may refer to significant results of de Figueiredo [6], Weinert, Desai, and Sidhu [12], and Sidhu and Weinert [9, 10]. However, the application of this technique to the optimal control problems with systems being governed by partial differential equations does not seem to be available in the literature until the author's recent paper (Chen [2]), where only the interpolation-type constraint is constrdered. In this paper, we extend the technique to solve the same problem as that studied in [2] with more complicated constraints, namely: inequality-constraint and smoothing constraint. We will provide complete solutions to these two problems with elegant expressions of the optimal solutions.

We remark that this research work was motivated by the significant work of Li [8], and that further generalizations of the author's result of [2] under interpolation-type constraints have already been made by the author and his collaborator in [3,5,7]. Besides optimal control theory and engineering, some applications of the new technique in signal processing may also be found in [4,5].

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2. Statement of problems

In this paper, we will consider the following two steady-state state-constrained minimum-energy optimal control problems:

Problem 1.

$$\underset{u \in L_2(D_a)}{\text{minimize}}\ F(u): \quad F(u) = \frac{1}{2}\iint_{D_a} u^2(x, y)\,dxdy \tag{2.1a}$$

subject to

$$\Delta w: = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = u(x, y) \tag{2.1b}$$

with

$$w(x, y)\,|_{\partial D_a} = \varphi(x, y) \tag{2.1c}$$

and

$$a_i \leqslant w(x_i, y_i) \leqslant \beta_i, \quad (x_i, y_i) \in D_a,\ i = 1, 2, \cdots, n, \tag{2.1d}$$

where

$$D_a = \{(x, y) \in R^2:\ x^2 + y^2 < a^2 < \infty\}, \tag{2.1e}$$

∂D_a is the boundary of D_a: $\partial D_a = \{(x, y) \in R^2: x^2 + y^2 = a^2\}$, $\varphi(x, y)$ is a given continuous function, and $\{z_i\}_{i=1}^n$ is a set of given scattered data.

We remark that the same problem with interpolation constraints, namely: all inequalities in (2.1d) become equalities, has been thoroughly studied in the author's recent paper [2] to which the interested reader is referred for more details.

We also remark that this mathematical model has the following simple physical interpretation: If the function $w(x, y)$ in (2.1b) is the displacement of an elastic membrane, then the problem is to find a closed-form optimal (i.e., with a minimum energy $F(u^*)$) distributed load $u^*(x, y)$ and the corresponding closed-form displacement function $w^*(x, y)$ such that $w^*(x, y)$ satisfies the simply supported condition (2.1c) on the boundary and the prescribed displacement conditions (2.1d) at n distinct interior points.

Problem 2

$$\underset{u \in L_2(D_a)}{\text{minimize}}\ F(u): \quad F(u) = \rho \sum_{i=1}^{k} (w(x_i, y_i) - z_i)^2 + \frac{1}{2}\iint_{D_a} u^2(x, y)\,dxdy \tag{2.2a}$$

subject to

$$\Delta w: = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = u(x, y) \tag{2.2b}$$

with

$$w(x, y)\,|_{\partial D_a} = \varphi(x, y) \tag{2.2c}$$

where $\rho \in (0, \infty)$ is a weight number, $\{z_i\}_{i=1}^k$ is a set of given scattered data, and other notations are the same as above.

The physical interpretation of Problem 2 is similar to that of Problem 1.

3. Reformulations of Problems 1 and 2

Using the technique developed in the author's paper [2], Problems 1 and 2 can be reformulated as follows: Let w_0 be the unique solution of the following Dirichlet problem:

$$\begin{cases} \Delta w = 0, \ (x, y) \in D_a \\ w|_{\partial D_a} = \varphi, \end{cases} \tag{3.1}$$

where the set D_a and function φ are defined as above. For Problem 1, let w_1 be the optimal solution of the following minimization problem:

$$\begin{cases} \underset{w \in W_0}{\text{minimize}} F_0(w): \ F_0(w) = \frac{1}{2}\iint_{D_a} (\Delta w)^2 \mathrm{d}x\mathrm{d}y, \\ w|_{\partial D_a} = 0, \\ a_i - w_0(x_i, y_i) < w(x_i, y_i) < \beta_i - w_0(x_i, y_i), \ i = 1, 2, \cdots, n, \\ W_0 = \{u \in L_2(D_a): u|_{\partial D_a} = 0\}, \end{cases} \tag{3.2}$$

where w_0 is obtained from (3.1). For Problem 2, let w_1 be the optimal solution of the following minimization problem:

$$\begin{cases} \underset{w \in W_0}{\text{minimize}} F_0(w): \ F_0(w) = \rho \sum_{i=1}^{k} (w(x_i, y_i) - (z_i - w_0(x_i, y_i)))^2 \\ \qquad\qquad + \frac{1}{2}\iint_{D_a} (\Delta w)^2 \mathrm{d}x\mathrm{d}y, \\ w|_{\partial D_a} = 0, \\ W_0 = \{w \in L_2(D_a): w|_{\partial D_a} = 0\}, \end{cases} \tag{3.3}$$

where again w_0 is obtained from (3.1). Then it can be easily verified that the optimal solutions u^* and w^* of Problems 1 and 2 are both given by

$$u^* = \Delta w^* = \frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2} \tag{3.4}$$

and

$$w^* = w_0 + w_1, \tag{3.5}$$

in which the only difference for the two problems is the term w_1. This will be carried out in the next section.

4. Optrimal solutions of Problems 1 and 2

As can be easily seen from the last section, the key step for solving problems 1 and 2 is to obtain closed-form solutions w_0 for both problems and different closed-form solutions w_1 for two different problems. This can be done as did in Chen [2] for the interpolation-constrained problem, and the results are given as follows: First, w_0 is obtained via the Poisson formula as

$$w_0(x, y) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t) \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - t) + r^2} \mathrm{d}t. \tag{4.1}$$

Secondly, it has been proved in Chen [1] that the space W_0 is a reproducing kernel Hilbert space with the kernel $K(r)-\widetilde{K}(r,\theta)$ in polar coordinates given by

$$K(r)=\frac{r^2}{8\pi}\ln r, \tag{4.2}$$

where $r=(x^2+y^2)^{1/2}$, $x=r\cos\theta$, $y=r\sin\theta$, and $\widetilde{K}(r,\theta):=\widetilde{K}(x,y)$ with

$$\begin{aligned}K(x,y)=&\frac{1}{2\pi}\int_0^{2\pi}K(t)\frac{a^2-r^2}{a^2-2ar\cos(\theta-t)+r^2}\mathrm{d}t\\&+\iint_{D_a}H(x,y;s,t)G(s,t)\mathrm{d}s\mathrm{d}t\end{aligned} \tag{4.3}$$

where

$$G(x,y)=\frac{1}{2\pi}\int_0^{2\pi}(\Delta K(t))\frac{a^2-r^2}{a^2-2ar\cos(\theta-t)+r^2}\mathrm{d}t$$

and

$$\begin{aligned}H(x,y;s,t)=&\frac{1}{2\pi}\{\ln((x-s)^2+(y-t)^2)^{1/2}\\&-\ln[\frac{(s^2+t^2)^{1/2}}{a}((x-\frac{a^2s}{s^2+t^2})^2+(y-\frac{a^2t}{s^2+t^2})^2)^{1/2}]\}.\end{aligned}$$

Then, for Problem 1, we have proved in [1] that the optimal solution w^* is given by

$$(\Delta^*\Delta)w^*=\sum_{j=1}^n c_jh(x-x_j,y-y_j), \tag{4.4}$$

where $h(x-x_j,y-y_j)$ are point-evaluation functionals, or equivalently,

$$w^*=w_0+\sum_{j=1}^n c_j[K(x-x_j,y-y_j)-\widetilde{K}(x-x_j,y-y_j)],$$

where the constants $\{c_j\}_{j=1}^n$ are determined by the following quadratic programming:

$$\min_{\{c_j\}}\frac{1}{2}\iint_{D_a}(\Delta w^*)^2, \tag{4.6}$$

subject to

$$\begin{aligned}a_i-w_0(x_i,y_i)<\sum_{j=1}^n c_j[&K(x_i-x_j,y_i-y_j)\\&-\widetilde{K}(x_i-x_j,y_i-y_j)]<\beta_i-w_0(x_i,y_i)\\&i=1,2,\cdots,n.\end{aligned}$$

For Problem 2, we need to solve the following problem in order to find the optimal solution w_1^*:

$$\begin{cases}\Delta^2w_1^*=\sum_{i=1}^k\mu_i(K(x-x_i,y-y_i)-\widetilde{K}(x-x_i,y-y_i))\\ w_1^*|_{\partial D_a}=0\\ \mu_i=\rho(z_i-w_0(x_i,y_i)-\langle w_1^*,K(x-x_i,y-y_i)-\widetilde{K}(x-x_i,y-y_i)\rangle_{w_0}),\end{cases} \tag{4.7}$$

where $i=1,\cdots,k$,

$$\langle f,g\rangle_{w_0}:=\frac{1}{2}\iint_{D_a}(\Delta f)(\Delta g)\mathrm{d}x\mathrm{d}y,$$

and other notations are the same as before. the proof of this resuli can be found in Chen [1].

Finally, we remark that Problems 1 and 2 may incorporate one more bound dary constraint of the form:

$$\frac{\partial w(x,y)}{\partial \vec{n}}\Big|_{\partial D_a}=\psi(x,y), \tag{4.8}$$

which is to be added to either (2.1c) or (2.2c), where $\vec{n}$ is the outer normal and $\psi(x,y)$ is a given continuous function. In this case, all conclusions hold except w_0 is now the unique solution of the following:

$$\begin{cases}\Delta^2 w=0, & (x,y)\in D_a\\ w|_{\partial D_a}=\varphi, & \dfrac{\partial w}{\partial \vec{n}}\Big|_{\partial D_a}=\psi,\end{cases}$$

and is given via a Poisson-like formula (cf. Tikhonov and Samarski [11]) by

$$w_0(x,y)=\frac{1}{2\pi a}(r^2-a^2)\left\{\frac{1}{2}\int_0^{2\pi}\frac{-\psi(t)}{a^2-2ar\cos(\theta-t)+r^2}dt+\int_0^{2\pi}\frac{\varphi(t)[a-r\cos(\theta-t)]}{[a^2-2ar\cos(\theta-t)+r^2]^2}dt\right\}.$$

The reader is referred to Chen [1,2] for more details for this discussion.

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一个稳态PDE系统的状态约束极小能最优控制问题

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摘　要

本文研究一个由偏微分方程描述的调和型稳态系统带不等式或光顺约束条件的极小能最优控制问题．基于由笔者新近发展的样条函数最优解技术，我们对所论问题可以给出完整的显式解析型最优解．问题的最优解可以通过一种优雅的再生核结构来表达．这种新技巧还可以用来对其他不同领域如图象和信号的最优再现等问题作统一的处理，体现出样条函数新理论的优越性．

这种样条函数技巧的前身只能解决由常微分方程描述的集中参数系统问题．有关这方面的历史发展和评述可参阅笔者下面的综合报告*．本文的结果是这一研究方向的新发展．

* 陈关荣，最优控制计算中的样条函数方　，应用数学与计算数学，1981年第5期17～25页．