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# Viscosity Method for the 2 × 2 Quasilinear Hyperbolic Conservation Laws *

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**Abstract**

By viscosity method, we prove the existence of global smooth solution to the system of 2×2 genuinely nonlinear hyperbolic conservation laws.

Consider the Cauchy problem of the systems of 2×2 conservation laws

$$U_t+F(U)_x=0, \quad (x,t)\in(-\infty,\infty)x(0,\infty), \tag{E}$$

$$U(x,0)=U_0(x), \quad x\in(-\infty,\infty) \tag{I}$$

where $U=(u,v)^T$, $F(U)=(f(u,v),g(u,v))^T$, $U_0(x)=(u_0(x),v_0(x))^T$, $f$ and $g$ are smooth in an open region $D_*$. (E) is assumed to be hyperbolic, i.e., $\nabla F(U)$ has real and distinct eigenvalues $\lambda<\mu$, and genuinely nonlinear in Lax's sense.

It is well known [1,2] that under the conditions (M), (C) and (V)( see below), the Cauchy problem (E), (I) has a global smooth solution. However, the methods used in [1,2] can not be used to get weak solutions. The intent of this paper[1)] is to make a first step to solve (E), (I) by viscosity method. This paper can be regarded as an extension of [4], in which (E) is the system of isentropic gas dynamics equations, and the condition (V) is removed.

Consider the system (E) with viscous terms, namely

$$U_t+F(U)_x=\varepsilon U_{xx}, \ \varepsilon>0, \quad (x,t)\in(-\infty,\infty)x[0,\infty). \tag{E}_E$$

Introduce Riemann invariants $(z,w)$. Thus, we can diagonalize $(E)_\varepsilon$ into

$$\begin{cases} z_t+\lambda z_x=\varepsilon z_{xx}-\varepsilon\nabla^2 zU_x^2, & (1)_1 \\ w_t+\mu w_x=\varepsilon w_{xx}-\varepsilon\nabla^2 wU_x^2, & (1)_2 \end{cases}$$

where $\nabla w\cdot r_\lambda=0$, $\nabla z\cdot r_\mu=0$; $r_\lambda$ and $r_\mu$ are the right characteristic vectors of $\nabla F$ with respect to the characteristic values $\lambda$ anv $\mu$; $\nabla f\equiv(f_u,f_v)$ for all smooth

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1) In detail, the intent of this paper has been described in [3].

*Dedicated to Professor L. C. Hsu. On the occasion of his 70-th Birthday.*

functions $f$. Since

$$\begin{cases} w_x = w_u u_x + w_v v_x \ , & (2)_1 \\ z_x = z_u u_x + z_v v_x \ , & (2)_2 \end{cases}$$

then

$$\begin{cases} u_x = A_1 w_x + A_2 z_x \ , & (3)_1 \\ v_x = B_1 w_x + B_2 z_x \ , & (3)_2 \end{cases}$$

where $A_1 = z_v/D$, $A_2 = -w_v/D$, $B_1 = -z_u/D$, $B_2 = w_u/D$, $D = w_u z_v - z_u w_v$. Without loss of generality, suppose the mapping $R$: $(u, v) \to (z, w)$ is one to one and smooth on $D_*$, $D \neq 0$. Substituting (3) into (1), by straight calculation, we have

$$\begin{cases} z_t + \lambda z_x = \varepsilon z_{xx} - \varepsilon(b_1 w_x^2 + b_2 z_x^2 + b_3 w_x z_x), & (4)_1 \\ w_t + \mu w_x = \varepsilon w_{xx} - \varepsilon(a_1 w_x^2 + a_2 z_x^2 + a_3 w_x z_x), & (4)_2 \end{cases}$$

where

$a_1 = A_1^2 w_{uu} + 2A_1 B_1 w_{uv} + B_1^2 w_{vv}$, $\quad a_2 = A_2^2 w_{uu} + 2A_2 B_2 w_{uv} + B_2^2 w_{vv}$,

$b_1 = A_1^2 z_{uu} + 2A_1 B_1 z_{uv} + B_1^2 z_{vv}$, $\quad b_2 = A_2^2 z_{uu} + 2A_2 B_2 z_{uv} + B_2^2 z_{vv}$,

$a_3 = 2A_1 A_2 w_{uu} + 2(A_1 B_2 + A_2 B_1) w_{uv} + 2B_1 B_2 w_{vv}$, $\quad b_3 = 2A_1 A_2 z_{uu} + 2(A_1 B_2 + A_2 B_1) z_{uv} + 2B_1 B_2 z_{vv}$.

Let

$$\varphi = e^{\alpha} w_x, \quad \theta = e^{\beta} z_x \ , \tag{5}$$

where $\alpha_z = \mu_z/(\mu - \lambda)$, $\beta_w = \lambda_w/(\lambda - \mu)$. Differentiating both sides of $(4)_2$ with respect to $x$, we have

$$\frac{\partial w_x}{\partial t} + \mu \frac{\partial w_x}{\partial x} = \varepsilon \frac{\partial^2 w_x}{\partial x^2} - \mu_w w_x^2 - \mu_z w_x z_x - \varepsilon(a_1 w_x^2 + a_2 z_x^2 + a_3 w_x z_x)_x \ . \tag{6}$$

By (5)

$$\frac{\partial w_x}{\partial t} = e^{-\alpha} \frac{\partial \varphi}{\partial t} - \alpha_t e^{-\alpha} \varphi, \tag{7$_1$}$$

$$\frac{\partial w_x}{\partial x} = e^{-\alpha} \frac{\partial \varphi}{\partial x} - \alpha_x e^{-\alpha} \varphi, \tag{7$_2$}$$

$$\frac{\partial^2 w_x}{\partial x^2} = e^{-\alpha} \frac{\partial^2 \varphi}{\partial x^2} - 2\alpha_x e^{-\alpha} \frac{\partial \varphi}{\partial x} + (\alpha_x^2 - \alpha_{xx}) e^{-\alpha} \varphi \tag{7$_3$}$$

Substituting (7) into (6), we have

$$\frac{\partial \varphi}{\partial x} + \mu \frac{\partial \varphi}{\partial x} = \varepsilon \frac{\partial^2 \varphi}{\partial x^2} + (\alpha_t + \mu \alpha_x)\varphi - e^{\alpha} \mu_w w_x^2 - \mu_z z_x \varphi + \varepsilon(\alpha_x^2 - \alpha_{xx})\varphi$$
$$- 2\varepsilon \alpha_x \varphi_x - e^{\alpha} \varepsilon (a_1 w_x^2 + a_2 z_x^2 + a_3 w_x z_x)_x \tag{8}$$

Since

$$\begin{aligned} \alpha_t + \mu \alpha_x &= \alpha_w(w_t + \mu w_x) + \alpha_z(z_t + \mu z_x) \\ &= \alpha_w(w_t + \mu w_x) + \alpha_z(z_t + \lambda z_x) + (\mu - \lambda)\alpha_z z_x \\ &= \varepsilon(\alpha_w w_{xx} + \alpha_z z_{xx}) + \mu_z z_x - \\ &\quad - \varepsilon[\alpha_w(a_1 w_x^2 + a_2 z_x^2 + a_3 w_x z_x) + \alpha_z(b_1 w_x^2 + b_2 z_x^2 + b_3 w_x z_x)]. \end{aligned} \tag{9}$$

Substituting (9) into (8), it is easy to get

$$\frac{\partial\varphi}{\partial t}+\mu\frac{\partial\varphi}{\partial x}=\varepsilon\frac{\partial^2\varphi}{\partial x^2}-e^{\alpha}\mu_w w_x^2-\varepsilon[2a_x\varphi_x+e^{\alpha}(a_1w_x^2+a_2z_x^2+a_3w_xz_x)_x]$$
$$+\varepsilon\varphi[(\alpha_w w_{xx}+\alpha_z z_{xx})-\alpha_w(a_1w_x^2+a_2z_x^2+a_3w_xz_x)$$
$$-\alpha_z(b_1w_x^2+b_2z_x^2+b_3w_xz_x)]+\varepsilon\varphi(\alpha_x^2-\alpha_{xx}).$$

Since

$$\alpha_x=\alpha_w w_x+\alpha_z z_x=e^{-\alpha}\alpha_w\varphi+e^{-\beta}\alpha_z\theta\quad, \qquad (1!$$
$$\alpha_{xx}=(e^{-\alpha}\alpha_w)_w e^{-\alpha}\varphi^2+[(e^{-\alpha}\alpha\ )_z e^{-\beta}+(e^{-\beta}\alpha_z)_w e^{-\alpha}]\varphi\theta$$
$$(e^{-\beta}\alpha_z)_z e^{-\beta}\theta^2+e^{-\alpha}\alpha_w\frac{\partial\varphi}{\partial x}+e^{-\beta}\alpha_z\frac{\partial\theta}{\partial x}, \qquad (11)_2$$
$$\beta_x=\beta_w w_x+\beta_z z_x=e^{-\alpha}\beta_w\varphi+e^{-\beta}\beta_z\theta. \qquad (11)_3$$

By (5), (7) and (11), we can rewrite (10) into

$$\frac{\partial\varphi}{\partial t}+\mu\frac{\partial\varphi}{\partial x}=\varepsilon\frac{\partial^2\varphi}{\partial x^2}-e^{-\alpha}u_w\varphi^2+\varepsilon f_1(\theta,\varphi)+\varepsilon f_2\varphi_x+\varepsilon f_3\theta_x, \qquad (12)$$

wh re

$$f_1(\theta,\omega)=(a_1\alpha_w-\alpha_{ww}-b_1\alpha_z-a_{1w}+\alpha_w)e^{-2\alpha}\varphi^3$$
$$+(a_3\beta_w+2a_1\alpha_z-b_3\alpha_z+2\alpha_w\alpha_z-a_{1z}-a_{3w}-2\alpha_{wz})e^{-\alpha-\beta}\varphi^2\theta+(2a_2\beta_z-a_{2z})e^{\alpha-3\beta}\theta^3$$
$$(2a_2\beta_w+a_3\alpha_z+a_3\beta_z-a_2\alpha_w-b_2\alpha_z+\alpha_z^2-\alpha_{zz}-a_{2w}-a_{3z})e^{-2\beta}\varphi\theta^2,$$
$$f_2(\theta,\varphi)=-2a_1e^{-\alpha}\varphi-a_3e^{-\beta}\theta-2\alpha_w\varphi e^{-\alpha}-2e^{-\beta}\alpha_z\theta,$$
$$f_3(\theta,\varphi)=-2a_2e^{\alpha-2\beta}\theta-a_3e^{-\beta}\varphi.$$

Similarly, we can prove

$$\frac{\partial\theta}{\partial t}+\lambda\frac{\partial\theta}{\partial x}=\varepsilon\frac{\partial^2\theta}{\partial x^2}-e^{-\beta}\lambda_z\theta^2+\varepsilon g_1(\theta,\varphi)+\varepsilon g_2(\theta,\varphi)\varphi_x+\varepsilon g_3(\theta,\varphi)\theta_x, \qquad (13)$$

where

$$g_1(\theta,\varphi)=(2b_1\alpha_w-b_{1w})e^{\beta-3\alpha}\varphi^3+(b_2\beta_z-b_{2z}-a_2\beta_w+\beta_z^2-\beta_{zz})e^{-2\beta}\theta^3$$
$$+(2b_1\alpha_z-b_{1z}-b_{3w}-a_1\beta_w-b_1\beta_z+\beta_w^2-\beta_{ww}+b_3\beta_w+b_3\alpha_w)e^{-2\alpha}\varphi^2\theta$$
$$+(2b_2\beta_w-b_{2w}-b_{3z}+b_3\alpha_z-a_3\beta_w+2\beta_w\beta_z-2\beta_{wz})e^{-\alpha-\beta}\varphi\theta^2,$$
$$g_2(\theta,\varphi)=-2b_1\varphi e^{-2\alpha+\beta}-b_3\theta e^{-\alpha},$$
$$g_3(\theta,\varphi)=-2\beta_w\varphi e^{-\alpha}-2\beta_z\theta e^{-\beta}-2b_2\theta e^{-\beta}-b_3\varphi e^{-\alpha}.$$

Our main result is the following theorem.

**Theorem I** Consider the Cauchy problem (E), (I), where $f$ and $g$ are smooth functions in an open region $D_*$. The matrix $\nabla F$ has two eigenvalues $\lambda$, $\mu$, satisfying $\lambda<\mu$, $\lambda_z>0$ and $\mu_w>0$ for all $(u,v)\in D_*$, i.e. (E) is strictly hyperbolic and genuinely nonlinear. If the initial data satisfy the following conditions

$$z_0'(x)\geq 0,\ w_0'(x)\ [illegible]\ 0, \qquad (M)$$
$$z_0(x),\quad w_0(x)\in C^3(-\infty,\infty), \qquad (C)$$
$$R^{-1}(D_1)\subset\subset D_* \qquad (V)$$

where $D_1=\{(z,w)\mid z_{0*}\leq z\leq z_0^*, w_{0*}\leq w\leq w_0^*\}$, $z_{0*}=\inf_x z_0(x)$, $z_0^*=\sup_x z_0(x)$, $w_{0*}=$

$\inf_x w_0(x)$, $w_0^* = \sup_x w_0(x)$, then the solutions of $(E)_\varepsilon$, (I) are uniformly bounded with respect to small $\varepsilon > 0$ in the strip $(-\infty, \infty)$ $[0, T)$, where $T > 0$ is any given constant. And these solutions uniformly converge to a smooth function as $\varepsilon$ tends to zero. The limit function is a smooth solution of (E), (I).

**Proof** By (V), $\max_{D_1} \mu < \infty$, $\min_{D_1} \lambda > -\infty$, then it is no harm to assume the initial data are constants outside a finite interval $[-X, X]$, where $X > 0$ is any fixed constant (see [4]).

Since $z_0(x)$ and $w_0(x)$ are bounded and smooth, by the conditions (C) and (M) we have $0 \le z_0'(x) \le l$, $0 \le w_0'(x) \le l$ and $|z_0''(x)|$, $|w_0''(x)| \le h$, where $l$ and $h$ are fixed constants.

For any small $\xi > 0$, let

$$D_\xi = \{(z, w) \mid z_{0*} - \xi < z < z_0^* + \xi,\ w_{0*} - \xi < w < w_0^* + \xi\}. \tag{14}$$

Since $R^{-1}(D_1) \subset\subset D_*$, then, for sufficiently small $\xi > 0$, $R^{-1}(D_\xi) \subset\subset D_*$.

We need the following lemma.

**Lemma 2** For any given $T > 0$, if $\varepsilon > 0$ is sufficiently small and

$$-\sqrt{\varepsilon c_1(e^t - 1)} < \varphi, \theta < L + \varepsilon(1 - e^{-t}),\ 0 \le t < T, \tag{15}$$

(cf (5)), then

$$(z, w) \in D_\xi,\ 0 \le t < T, \tag{16}$$

where $(z, w)$ are solutions of $(E)_\varepsilon$, (I); $c_1$ is positive constant which depends only on $D_\xi$, $F, t$, and $L$, $L = \sup_x \{|\varphi_0(x)|, |\theta_0(x)|\}$.

**Proof** Let

$$T_1 = \sup\{t \mid (z(\cdot, t), w(\cdot, t) \in D_\xi\}. \tag{17}$$

Obviously, in order to prove the lemma, it is sufficient to prove $T_1 = T$. We now prove $T_1 = T$ by contradiction. Suppose $T_1 < T$, then there exists a point $(x_1, T_1)$ satisfying $z(x_1, T_1) = z_0 - \xi$ (or $w(x_1, T_1) = w_{0*} - \xi$), or $z(x_1, T_1) = z_0^* + \xi$ (or $w(x_1, T_1) = w_0^* + \xi$). We only need to examine these two possible cases.

(I) If $z(x_1, T_1) = z_{0*} - \xi$ (or $w(x_1, T_1) = w_{0*} - \xi$), $T_1 < T$. Let $\beta_* = \inf_{D_\xi} \beta$, $\mu^* = \sup_{D_\xi} \mu$, $\lambda_* = \inf_{D_\xi} \lambda$, by (15), we have

$$-e^{-\beta_*}\sqrt{\varepsilon c_1(e^t - 1)} < z_x < Le^{-\beta_*} + \varepsilon e^{-\beta_*}(1 - e^{-t}),\ 0 \le t < T_1. \tag{18}$$

Thus

$$\begin{aligned} z(x_1, T_1) &\ge z_{0*} - e^{-\beta_*}\sqrt{\varepsilon c_1(e^{T_1} - 1)}\,(2X + T_1(\mu^* - \lambda_*)) \\ &\ge z_{0*} - e^{-\beta_*}\sqrt{\varepsilon c_1 e^{T_1}}\,(2X + T_1(\mu^* - \lambda_*)). \end{aligned} \tag{19}$$

Let $0 < \varepsilon < \xi^2 e^{2\beta_*}[c_1 e^T(2X + T(\mu^* - \lambda_*))^2]^{-1}$, by (19) we have

$$z(x_1, T_1) > z_{0*} - \xi. \tag{20}$$

But we have assumed $z(x_1, T_1) = z_{0*} - \xi$, then we get a contradiction.

(II) If $z(x_1, T_1) = z_0^* + \xi$ (or $w(x_1, T_1) = w_0^* + \xi$), $T_1 < T$. Since $\lim_{x \to \infty}(z(x_1, T_1)$,

$w(x_1, T_1)) = (z_0(X), w_0(X)) \le (z_0^*, w_0^*)$, then following the proof in (I), we can introduce a contradiction. ∎

According to Lemma 2, it is easy to see that the major step of the proof is to prove (15). To this end, we let

$$T_2 = \sup\{t \mid -\sqrt{\varepsilon c_1(e^t - 1)} < \varphi(\cdot, t), \theta(\cdot, t) < L + \varepsilon(1 - e^{-t})\}, \tag{21}$$

where the constant $c_1$ will be defined later, which is independent of $\varepsilon$. In order to prove (15), it is sufficient to prove $T_2 = T$ as $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0$ is a constant which depends only on $F$, $D_\xi$, $L$ and $T$.

To do this, we need the following Lemma 3.

**Lemma 3** If (15) holds, then

$$|\varphi_x|, \ |\theta_x| < c_2, \quad 0 \le t < T, \tag{22}$$

where $c_2$ will be defined later, which is independent of $\varepsilon$.

We will prove the lemma later.

**Case 1** There exists $(x_2, T_2)$ such that $\varphi(x_2, T_2) = L + \varepsilon(1 - e^{-T_2})$ (or $\theta(x_2, T_2) = L + \varepsilon(1 - e^{-T_2})$), $T_2 < T$. It implies $\left.\frac{\partial \varphi}{\partial x}\right|_{(x_2, T_2)} = 0$, $\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_{(x_2, T_2)} \le 0$. By (12), we have

$$\begin{aligned}\left.\frac{\partial \varphi}{\partial t}\right|_{(x_2, T_2)} &\le -(e^{-\alpha}\mu_w)|_{(x_2, T_2)}\varphi^2 + \varepsilon n(1 + 2c_2) \\ &\le -m(L + \varepsilon(1 - e^{-T_2}))^2 + \varepsilon n(1 + 2c_2) < 0\end{aligned} \tag{23}$$

as $0 < \varepsilon < mL^2(1 + 2c_2)^{-1} n^{-1}$, where $m = \inf\limits_{D_\xi}\{e^{-\alpha}\mu_w, e^{-\beta}\mu_z\} > 0$, $n = \sup\limits_{D_\xi}\{|g_i|, |f_i|, i = 1, 2, 3\}$. Obviously, $m$ and $n$ are constants which are independent of $\varepsilon$ and $T_2$. On the other hand, by the definition of $T_2$,

$$\begin{aligned}\left.\frac{\partial \varphi}{\partial t}\right|_{(x_2, T_2)} &= \lim_{t \to T_2 - 0} \frac{\varphi(x_2, T_2) - \varphi(x_2, t)}{T_2 - t} \\ &\ge \lim_{t \to T_2 - 0} \frac{[L + \varepsilon(1 - e^{-T_2})] - [L + \varepsilon(1 - e^{-t})]}{T_2 - t} = \varepsilon e^{-T_2} > 0.\end{aligned} \tag{24}$$

(24) contradicts (23), thus there is no $T_2 < T$ such that $\varphi(x_2, T_2) = L + \varepsilon(1 - e^{-T_2})$. Similarly, there is no $T_2 < T$ such that $\theta(x_2, T_2) = L + \varepsilon(1 - e^{-T_2})$.

**Case 2** There exists $(x_2, T_2)$ such that $\varphi(x_2, T_2) = -\sqrt{\varepsilon c_1(e^{T_2} - 1)}$ (or $\theta(x_2, T_2) = -\sqrt{\varepsilon c_1(e^{T_2} - 1)}$). It implies that $\left.\frac{\partial \varphi}{\partial x}\right|_{(x_2 T_2)} = 0$, $\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_{(x_2, T_2)} \ge 0$. Thus by (12), we have

$$\left.\frac{\partial \varphi}{\partial x}\right|_{(x_2, T_2)} \ge -M(-\sqrt{\varepsilon c_1(e^{T_2} - 1)})^2 - \varepsilon n(1 + 2c_2) = -\varepsilon c_1 M e^{T_2} + \varepsilon[M c_1 - \varepsilon n(1 + 2c_2)], \tag{25}$$

where $M = \sup\limits_{D_\xi}\{e^{-\alpha}\mu_w, e^{-\beta}\lambda_z\}$. Let $c_1 > M^{-1} n(1 + 2c_2)$, by (25), we have

$$\left.\frac{\partial \varphi}{\partial t}\right|_{(x_2, T_2)} \ge -\varepsilon c_1 M e^{T_2}. \tag{26}$$

On the other hand, by the definition of $T_2$,

$$\left.\frac{\partial\varphi}{\partial t}\right|_{(x_2,T_2)}=\lim_{t\to T_2-0}\frac{\varphi(x_2,T_2)-\varphi(x_2,t)}{T_2-t}$$
$$\leq\lim_{t\to T_2-0}\frac{-\sqrt{\varepsilon c_1(e^{T_2}-1)}+\sqrt{\varepsilon c_2(e^{t}-1)}}{T_2-t}$$
$$=-\frac{1}{2}\varepsilon^{\frac{1}{2}}c_1^{\frac{1}{2}}e^{T_2}(e^{T_2}-1)^{-\frac{1}{2}}. \tag{27}$$

When $0<\varepsilon<[4M^2c_1(e^T-1)]^{-1}$, by (27) we have

$$\left.\frac{\partial\varphi}{\partial t}\right|_{(x_2,T_2)}<-\varepsilon c_1 M e^{T_2}. \tag{28}$$

(28) contradicts (26), thus there is no $(x_2, T_2)$, $T_2<T$, such that $\varphi(x_2,T_2)=-\sqrt{\varepsilon c_1(e^{T_2}-1)}$. Similarly, there is no $(x_2,T_2)$, $T_2<T$, such that $\theta(x_2,T_2)=-\sqrt{\varepsilon c_1(e^{T_2}-1)}$.

The contradictions imply $T_2=T$.

Finally, we only need to prove Lemma 3.

**Proof of Lemma 3** Differentiating (12), (13) with respect to $x$, by straight calculation, we have

$$\frac{\partial\varphi_x}{\partial t}+\mu\frac{\partial\varphi_x}{\partial x}=\varepsilon\frac{\partial^2\varphi_x}{\partial x^2}+f_4+f_5\varphi_x+\varepsilon f_6+\varepsilon\frac{\partial}{\partial x}(f_2\varphi_x)+\varepsilon\frac{\partial}{\partial x}(f_3\theta_x), \tag{29}$$
$$\frac{\partial\theta_x}{\partial t}+\lambda\frac{\partial\theta_x}{\partial x}=\varepsilon\frac{\partial^2\theta_x}{\partial x^2}+g_4+g_5\theta_x+\varepsilon g_6+\varepsilon\frac{\partial}{\partial x}(g_2\varphi_x)+\varepsilon\frac{\partial}{\partial x}(g_3\theta_x), \tag{30}$$

where

$$f_4=-(e^{-\alpha}\mu_w)_w e^{-\alpha}\varphi^3-(e^{-\alpha}\mu_w)_z e^{-\beta}\varphi^2\theta,\quad f_5=-3e^{-\alpha}\mu_w\varphi-\mu_z e^{-\beta}\theta,$$
$$f_6=f_{1w}e^{-\alpha}\varphi+f_{1z}e^{-\beta}\theta,\quad g_4=-(e^{-\beta}\lambda_z)_w e^{-\alpha}\varphi\theta^2-(e^{-\beta}\lambda_z)_z e^{-\beta}\theta^3,$$
$$g_5=-3e^{-\beta}\lambda_z\theta-\lambda_w e^{-\alpha}\varphi,\qquad g_6=g_{1w}e^{-\alpha}\varphi+g_{1z}e^{-\beta}\theta.$$

Let $M(t)=\sup\limits_{\pi_t}\{|\varphi_x|, |\theta_x|\}$, where $\pi_t=\{(x,\tau)|x\in\boldsymbol{R}, 0\leq\tau<t\}$, $N=\sup\limits_{D_\xi}\{|f_{i+3}|, |g_{i+3}|, i=1,2,3\}$. Obviously, $N$ is a constant which depends only on $D_\xi$, $F$, $L$ and $T$.

When $0<\varepsilon<\min\{L, 4L^2c_1^{-1}(e^T-1)^{-1}\}$, by (15) we have $|\varphi|$, $|\theta|<2L$. We now prove (22) by contradiction. Let

$$T_3=\sup\{t|M(t)<c_2\}. \tag{31}$$

Suppose $T_3<T$, and we divide the strip $(-\infty,\infty)x[0,T_3]$ into small strips $(-\infty,\infty)x[t_i,t_{i+1})$, $t_i=i\varepsilon$, $i=0,1,2,\cdots,[T_3\varepsilon^{-1}]-1$, here, it is no harm to assume $T_3\varepsilon^{-1}$ is an integer. For any given point $(x_0,t_0)$, $t_i\leq t_0\leq t_{i+1}$, we rewrite (29) into

$$\frac{\partial\varphi_x}{\partial t}+\mu(x_0,t_0)\frac{\partial\varphi_x}{\partial x}=\varepsilon\frac{\partial^2\varphi_x}{\partial x^2}+f_4+f_5\varphi_x+\varepsilon f_6+\varepsilon\frac{\partial}{\partial x}(f_2\varphi\ )$$
$$+\varepsilon\frac{\partial}{\partial x}(f_3\theta_x)+(\mu(x_0,t_0)-\mu(x,t))\frac{\partial\varphi_x}{\partial x}. \tag{32}$$

Define a coordinate transformation $t=t$, $x=y+\mu(x_0,t_0)(t-t_0)$. Let $\tilde{w}(y,t)=$

$w(x(y,t),t)$, then

$$\frac{\partial w(y,t)}{\partial t}=\frac{\partial w(x,t)}{\partial t}+\mu(x_0,t_0)\frac{\partial w(x,t)}{\partial x},\quad \frac{\partial w(y,t)}{\partial y}=\frac{\partial w(x,t)}{\partial x},$$

where $w$ is any smooth function. By (32), we have

$$\frac{\partial\varphi_y}{\partial t}=\varepsilon\frac{\partial^2\varphi_y}{\partial y^2}+f_4+f_5\varphi_y+\varepsilon f_6+\varepsilon\frac{\partial}{\partial y}(\widetilde{f_2}\widetilde{\varphi_y})$$

$$+\varepsilon\frac{\partial}{\partial y}(f_3\theta_y)+(\mu(y_0,t_0)-\mu(y,t))\frac{\partial\widetilde{\varphi}_y}{\partial y},\tag{33}$$

where $\widetilde{\varphi}(y,t)=\varphi(x(y,t),t)$, $\widetilde{f}_i(y,t)=f_i(x(y,t),t)$, $i=2,3,4,5,6$, $\widetilde{\theta}(y,t)=\theta(x(y,t),t)$, $\widetilde{\mu}(y,t)=\mu(x(y,t),t)$, $\widetilde{u}(y_0,t_0)=\mu(x_0,t_0)$. Hereafter, we omit "~" for simplicity in printing.

If we let $\varphi_y(y,t_j)$ be the initial data, then by the equation (33), we have

$$\varphi_y(y_0,t_0)=\int_{\infty}^{\infty}\frac{\varphi_y(\xi,t_i)'}{2\sqrt{\varepsilon\pi(t_0-t_i)}}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-t_i)}\}d\xi$$

$$+\frac{1}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{f_4}{2\sqrt{\varepsilon(t_0-\tau)}}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}d\xi d\tau$$

$$+\frac{1}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{f_5\varphi_\xi}{2\sqrt{\varepsilon(t_0-\tau)}}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}d\xi d\tau$$

$$+\frac{\varepsilon}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{f_6}{2\sqrt{\varepsilon(t_0-\tau)}}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}d\xi d\tau$$

$$+\frac{\varepsilon}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{\partial/\partial\xi(f_2\varphi_\xi)}{2\sqrt{\varepsilon(t_0-\tau)}}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}d\xi d\tau$$

$$+\frac{\varepsilon}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{\partial/\partial\xi(f_3\theta_\xi)}{2\sqrt{\varepsilon(t_0-\tau)}}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}d\xi d\tau$$

$$+\frac{1}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{(\mu(y_0,t_0)-\mu)\varphi_{\xi\xi}}{2\sqrt{\varepsilon(t_0-\tau)}}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}d\xi d\tau$$

Thus

$$|\varphi_y(y_0,t_0)|\le M(t_i)+N(t_0-t_i)+N\int_{t_i}^{t_0}M(\tau)d\tau+\varepsilon N(t_0-t_i)+I,\tag{34}$$

where

$$I=\frac{\varepsilon}{\sqrt{\pi}}\left|\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{\partial/\partial\xi(f_2\varphi_\xi)}{2\sqrt{\varepsilon(t_0-\tau)}}\exp\{-\frac{(y_0-\xi)^2}{\varepsilon(t_0-\tau)}\}d\xi d\tau\right|$$

$$+\frac{\varepsilon}{\sqrt{\pi}}\left|\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{\partial/\partial\xi(f_3\theta_\xi)}{2\sqrt{\varepsilon(t_0-\tau)}}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}d\xi d\tau\right|$$

$$+\frac{1}{\sqrt{\pi}}\left|\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{(\mu-\mu_0(y_0,t_0))\varphi_{\xi\xi}}{2\sqrt{\varepsilon(t_0-\tau)}}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}d\xi d\tau\right|.$$

$$I\le\frac{\varepsilon}{\sqrt{\pi}}\left|\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{2f_2\varphi_\xi(y_0-\xi)}{(2\sqrt{\varepsilon(t_0-\tau)})^3}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t-\tau)}\}d\xi d\tau\right|$$

$$+\frac{\varepsilon}{\sqrt{\pi}}\left|\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{2f_3\theta_\xi(y_0-\xi)}{(2\sqrt{\varepsilon(t_0-\tau)})^3}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}d\xi d\tau\right|$$

$$+\frac{1}{\sqrt{\pi}}\left|\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{\mu_\xi\varphi_\xi}{2\sqrt{\varepsilon(t_0-\tau)}}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}\,d\xi d\tau\right|$$

$$+\frac{1}{\sqrt{\pi}}\left|\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{2(\mu-\mu(y_0,t_0))\varphi_\xi(y_0-\xi)}{(\sqrt{\varepsilon(t_0-\tau)})^3}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}\,d\xi d\tau\right|$$

$$\leq\frac{2n\varepsilon}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{M(\tau)|y_0-\xi|}{(2\sqrt{\varepsilon(t_0-\tau)})}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}\,d\xi d\tau$$

$$+\frac{2n\varepsilon}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{M(\tau)|y_0-\xi|}{(2\sqrt{\varepsilon(t_0-\tau)})^3}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}\,d\xi d\tau$$

$$+\frac{\varepsilon}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{|\mu_\xi|M(\tau)}{(2\sqrt{\varepsilon(t_0-\tau)})}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}\,d\xi d\tau$$

$$+\frac{2}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{|\mu_\xi|M(\tau)(y_0-\xi)^2}{(2\sqrt{\varepsilon(t_0-\tau)})^3}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}\,d\xi d\tau$$

$$+\frac{2}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{|\mu_t|M(\tau)(t_0-\tau)(y_0-\xi)}{(2\sqrt{\varepsilon(t_0-\tau)})^3}\exp\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\}\,d\xi d\tau\,. \tag{35}$$

Since

$$\mu_y=\mu_w e^{-\alpha}\varphi+\mu_z e^{-\beta}\theta,$$

$$\mu_t=u_w w_t+\mu_z z_t=\mu_w(\varepsilon w_{xx}-\varepsilon\nabla^2 wU_x^2-\mu w_x)+\mu_z(\varepsilon z_{xx}-\varepsilon\nabla^2 zU_x^2-\lambda z_x),$$

then without loss of generality, we can suppose that $|\mu_y|\leq N$ and $|\mu_t|\leq N$, $0\leq t<T_3$ hold for the same constant $N$ defined above. Let

$$r=\frac{y_0-\xi}{2\sqrt{\varepsilon(t_0-\tau)}}$$

then by (35) we have

$$I\leq\frac{2n\varepsilon}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{|r|M(\tau)}{2\sqrt{\varepsilon(t_0-\tau)}}e^{-r^2}dy\,d\tau+\frac{2n\varepsilon}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{|r|M(\tau)}{2\sqrt{\varepsilon(t_0-\tau)}}e^{-r^2}dy\,d\tau+N\varepsilon\int_{t_i}^{t_0}M(\tau)\,d\tau$$

$$+\frac{2N}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}M(\tau)r^2e^{-r^2}dr\,d\tau+\frac{2N}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{\sqrt{t_0-\tau}\,M(\tau)|r|}{2\sqrt{\varepsilon}}e^{-r^2}dr\,d\tau$$

$$\leq\frac{n\varepsilon}{\sqrt{\pi}}\frac{2(t_0-\tau)^{1/2}}{2\sqrt{\varepsilon}}M(t_0)+\frac{n\varepsilon}{\sqrt{\pi}}\frac{2(t_0-\tau)^{1/2}}{\sqrt{\pi}}M(t_0)+N\varepsilon\int_{t_i}^{t_0}M(\tau)\,d\tau$$

$$+N\int_{t_i}^{t_0}M(\tau)\,d\tau+\frac{N}{\sqrt{\pi\varepsilon}}\frac{2}{3}(t_0-\tau)^{3/2}M(t_0)$$

$$\leq\frac{4n}{\sqrt{\pi}}\varepsilon M(t_0)+N(1+\varepsilon)\int_{t_i}^{t_0}M(\tau)\,d\tau+\frac{2N\varepsilon}{3\sqrt{\pi}}M(t_0)$$

$$<(2n+N)\varepsilon M(t_0)+2N\int_{t_i}^{t_0}M(\tau)\,d\tau, \tag{36}$$

(36) holds for $0<\varepsilon<1$. Substituting (36) into (34), we have

$$|\varphi_y(y_0,t_0)|\leq M(t_i)+N(1+\varepsilon)(t_0-t_i)+3N\int_{t_i}^{t_0}M(\tau)\,d\tau+(2n+N)\varepsilon M(t_0). \tag{37}$$

Let $d_1=N(1+\varepsilon)$, $d_2=3N$, $d_3=2n+N$, then

$$|\varphi_y(y_0,t_0)|\le M(t_i)+d_1(t_0-t_i)+d_2\int_{t_i}^{t_0}M(\tau)\,d\tau+d_3\varepsilon M(t_0). \tag{38}$$

Similarly, we can prove

$$|\theta_y(y_0,t_0)|\le M(t_i)+d_1(t_0-t_i)+d_2\int_{t_i}^{t_0}M(\tau)\,d\tau+d_3\varepsilon M(t_0). \tag{39}$$

Since $(y_0,t_0)$ is arbitrary, then for any $t\in(t_i,t_{i+1}]$, we have

$$M(t)\le M(t_i)+d_1(t-t_i)+d_2\int_{t_i}^{t}M(\tau)\,d\tau+d_3\varepsilon M(t). \tag{40}$$

That is

$$(1-d_3\varepsilon)M(t)-d_2\int_{t_i}^{t}M(\tau)\,d\tau\le M(t_i)+d_1(t-t_i),$$

or

$$\left(\exp\{-\frac{d_2}{1-d_3\varepsilon}t\}\int_{t_i}^{t}M(\tau)\,d\tau\right)_t\le\frac{M(t_i)+d_1(t-t_i)}{1-d_3\varepsilon}\exp\{-\frac{d_2}{1-d_3\varepsilon}t\}. \tag{41}$$

Integrating both sides of (41) from $t_i$ to $t_{i+1}$, by straight calculation, we have

$$M(t_{i+1})+\frac{d_1}{d_2}\le\frac{1}{1-d_3\varepsilon}\exp\{\frac{d_2}{1-d_3\varepsilon}(t_{i+1}-t_i)\}\left(M(t_i)+\frac{d_1}{d_2}\right). \tag{42}$$

By (42) and the above proof, we have

$$M(T_3)+\frac{d_1}{d_2}\le(1-d_3\varepsilon)^{-T_3\varepsilon^{-1}+1}\left(M(0)+\frac{d_1}{d_2}\right)\exp\{\frac{d_2}{1-d_3\varepsilon}T_3\}.$$

Since $\lim\limits_{\varepsilon\to0}d_1=N$, $\lim\limits_{\varepsilon\to0}d_2=3N$, $\lim\limits_{\varepsilon\to0}d_3=2n+N$, $\lim\limits_{\varepsilon\to0}(1-d_3\varepsilon)^{-T\varepsilon^{-1}+1}=\exp\{(2n+N)T_3\}$, then there exists $\varepsilon_0>0$ which depends only on $D_\xi$, $F$, $L$ and $T$, such that if $0<\varepsilon<\varepsilon_0$, then

$$M(T_3)<\left(L+\frac{N}{3N}\right)\exp\{2(n+2N)T_3\}<(L+1)\exp\{2(n+2N)T_3\}\le c_2, \tag{43}$$

where $c_2=(L+1)\exp\{2(n+2N)T\}$. (43) contradicts $T_3<T$. Thus, (22) holds for $0\le t<T$. ■

The remainder of the proof is standard and we omit it. ■

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# 2×2拟线性双曲型守恒律组的粘性消失法

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考虑2×2严格双曲型守恒律组（E），它是在Lax意义下真正非线性的，带有初始条件（I）。众所周知，在条件（M），（C），（V）下，初值问题（E）、（I）存在整体光滑解，（参看文[1,2]）。然而在文[1,2]中所采用的方法本质地用来求广义解。本文是用粘性消失法证明文[1]的结果。我们把这个结果看作用粘性消失法求（E）、（I）的广义解的第一步。本文也可以看作文[4]的某种推广。在文[4]中，（E）是在Lagrange坐标下均熵气体动力学方程组，但无需条件（V），也是用粘性消失法求得光滑解。