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The Exterior Tricomi and Frankl Problem*

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Abstract

F. G. Tricomi (1923—), S. Gellerstedt (1935—), F. I. Frankl (1945—), A. V. Bitsadze and M. A. Lavrentiev (1950—), M. H. Protter (1953—) and most of the recent workers in the field of mixed type boundary value problems have considered only one parabolic line of degeneracy. The problem with more than one parabolic line of degeneracy becomes more complicated. The above researchers and many others have restricted their attention to the Chaplygin equation: $K(y)\cdot u_{xx}+u_{yy}=f(x,y)$ and not considered the "generalized Chaplygin equation:" $Lu=K(y)\cdot u_{xx}+u_{yy}+r(x,y)\cdot u=f(x,y)$ because of the difficulties that arise when r: =non-trivial ($\neq 0$). Also it is unusual for anyone to study such problems in a doubly connected region. In this paper I consider a case of this type with two parabolic lines of degeneracy, r: =non-trivial ($\neq 0$), in a doubly connected region, and such that boundary conditions are prescribed only on the "exterior boundary" of the mixed domain, and I obtain uniqueness results for quasiregular solutions of the characteristic and non-characteristic Problem by applying the b, c energy integral method in the mixed domain.

The Exterior Tricomi Problem

Consider

$$(+)\qquad Lu=K(y)\cdot u_{xx}+u_{yy}+r(x,y)\cdot u=f(x,y),\quad K\in C^2(\cdot),\ r\in C^1(\cdot),\ f\in C^0(\cdot),$$

and such that

$$K=K(y)>0 \text{ for } y<0 \text{ and } y>1,$$
$$:=0 \text{ for } y=0 \text{ and } y=1, \text{ and}$$
$$:<0 \text{ for } 0<y<1.$$

Consider a mixed domain D which is doubly connected, contains the two parabolic arcs: A_1B_1, A_2B_2, with end points $A_1=(-1,1)$, $B_1=(1,1)$, $A_2=(-1,0)$, $B_2=(1,0)$, and has boundary

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$$\partial D = \mathrm{Ext}(D) \cup \mathrm{Int}(D),$$

Ex(D): exterior boundary of D: $=\Gamma_0 \cup \Gamma_0' \cup \Gamma_2 \cup \Gamma_2' \cup \Delta_1 \cup \Delta_1'$, and

Int(D): interior boundary of D: $=\Gamma_1 \cup \Gamma_1' \cup \Delta_2 \cup \Delta_2'$,

with boundary curves:

Γ_0: "elliptic arc" for $y>1$ connecting points: A_1, B_1,

Γ_0' : "elliptic arc" for $y<0$ connecting points: A_2, B_2,

Γ_1 : characteristic for $0<y<1$, $0<x<1$ emanating from point:

$o_1=(0,1)$:

$$:\ \int_0^x \mathrm{d}x = -\int_1^y \sqrt{-K}\cdot \mathrm{d}y\ , \quad \text{or} \quad \Gamma_1:\ x = -\int_1^y \sqrt{-K(t)}\cdot \mathrm{d}t\ ,$$

Γ_1' : characteristic for $0<y<1$, $0<x<1$ emanating from point:

$o_2=(0,0)$:

$$:\ \int_0^x \mathrm{d}x = \int_0^y \sqrt{-K}\cdot \mathrm{d}y\ , \quad \text{or} \quad \Gamma_1':\ x = \int_0^y \sqrt{-K(t)}\cdot \mathrm{d}t\ ,$$

Γ_2 : characteristic for $0<y<1$, $0<x<1$ emanating from point:

$B_1=(1,1)$:

$$:\ \int_1^x \mathrm{d}x = \int_1^y \sqrt{-K}\cdot \mathrm{d}y\ , \quad \text{or} \quad \Gamma_2:\ x = \int_1^y \sqrt{-K(t)}\cdot \mathrm{d}t + 1\ ,$$

Γ_2' : characteristic for $0<y<1$, $0<x<1$ emanating from point:

$B_2=(1,0)$:

$$:\ \int_1^x \mathrm{d}x = -\int_0^y \sqrt{-K}\cdot \mathrm{d}y, \quad \text{or} \quad \Gamma_2':\ x = -\int_0^y \sqrt{-K(t)}\cdot \mathrm{d}t + 1,$$

Δ_1 : characteristic for $0<y<1$, $-1<x<0$ emanating from point:

$A_1=(-1,1)$:

$$:\ \int_{-1}^x \mathrm{d}x = -\int_1^y \sqrt{-K}\cdot \mathrm{d}y, \quad \text{or} \quad \triangle_1:\ x = -\int_1^y \sqrt{-K(t)}\cdot \mathrm{d}t - 1,$$

Δ_1' : characteristic for $0<y<1$, $-1<x<0$ emanating from point:

$A_2=(-1,0)$:

$$:\ \int_{-1}^x \mathrm{d}x = \int_0^y \sqrt{-K}\cdot \mathrm{d}y, \quad \text{or} \quad \triangle_1':\ x = \int_0^y \sqrt{-K(t)}\cdot \mathrm{d}t - 1,$$

Δ_2: characteristic for $0<y<1$, $-1<x<0$ emanating from point:

$o_1=(0,1)$:

$$:\ \int_0^x \mathrm{d}x = \int_1^y \sqrt{-K}\cdot \mathrm{d}y, \quad \text{or} \quad \triangle_2:\ x = \int_1^y \sqrt{-K(t)}\cdot \mathrm{d}t\ ,$$

Δ_3' : characteristic for $0<y<1$, $-1<x<0$ emanating from point:

$o_2=(0,0)$:

$$:\ \int_0^x \mathrm{d}x = -\int_0^y \sqrt{-K}\cdot \mathrm{d}y, \quad \text{or} \quad \triangle_2':\ x = -\int_0^y \sqrt{-K(t)}\cdot \mathrm{d}t\ .$$

Besides,

$$D=G_1\cup G_1'\cup G_2\cup G_2'\cup(A_1B_1)\cup(A_2B_2),$$

Where

G_1: upper elliptic region: $=\{(x,y)\in D,\ |x|<1,\ y>1\}$

G_1': lower elliptic region: $=\{(x,y)\in D,\ |x|<1,\ y<0\}$

G_2: right-hand side hyperbolic region: $=\{(x,y)\in D,\ 0<x<1,\ 0<y<1\}$

G_2': left-hand side hyperbolic region: $=\{(x,y)\in D,\ -1<x<0,\ 0<y<1\}$

with boundary

$\partial G_1:=\Gamma_0\cup(A_1B_1)$, $\partial G_1':=\Gamma_0'\cup(B_2A_2)$,

$\partial G_2:=\Gamma_1\cup\Gamma_1'\cup\Gamma_2\cup\Gamma_2'\cup(B_1O_1)\cup(O_2B_2)$,

$\partial G_2':=\triangle_1\cup\triangle_1'\cup\triangle_2\cup\triangle_2'\cup(O_1A_1)\cup(A_2O_2)$.

The above characteristic curves intersect at the following points:

$\Gamma_1\cap\Gamma_1'=P_1$, $\Gamma_2\cap\Gamma_2'=P_2$ for $0<y<1$ and $0<x<1$, and

$\triangle_1\cap\triangle_1'=P_1'$, $\triangle_2\cup\triangle_2'=P_2'$ for $0<y<1$ and $-1<x<0$.

Besides, assume boundary conditions

$$(++)\quad\begin{cases}u=\phi_1(s)\ \text{on}\ \Gamma_0, & u=\phi_2(s)\ \text{on}\ \Gamma_0'\\ u=\psi_1(x)\ \text{on}\ \Gamma_2, & u=\psi_2(x)\ \text{on}\ \Gamma_2'\\ u=\phi_3(x)\ \text{on}\ \triangle_1, & u=\psi_4(x)\ \text{on}\ \triangle_1'\end{cases}$$

(i.e.: u: = continuous prescribed values on Ext(D)):

The Exterior Tricomi Proelem, or Problem (ET):

Consists in finding a function $u=u(x,y)$ which satisfies equation (+) and boundary conditions (++).

A New Uniqueness Theorem

Assume the above-mentioned domain $D\subset R^2$, and the conditions

(R_1): $\quad r\leqq 0$ on Int(D)

$$(R_2):\quad\begin{cases}x\cdot dy-(y-1)\cdot dx\geqq 0 & \text{on}\ \Gamma_0\\ x\cdot dy-y\cdot dx>0 & \text{on}\ \Gamma_0'\end{cases}$$

"star-likedness"

$$(R_3):\quad\begin{cases}2\cdot r+x\cdot r_x+(y-1)\cdot r_y<0 & \text{in}\ G_1\\ r+x\cdot r_x\leqq 0 & \text{in}\ G_2\cup G_2'\\ 2\cdot r+x\cdot r_x+y\cdot r_y\leqq 0 & \text{in}\ G_1'\end{cases}$$

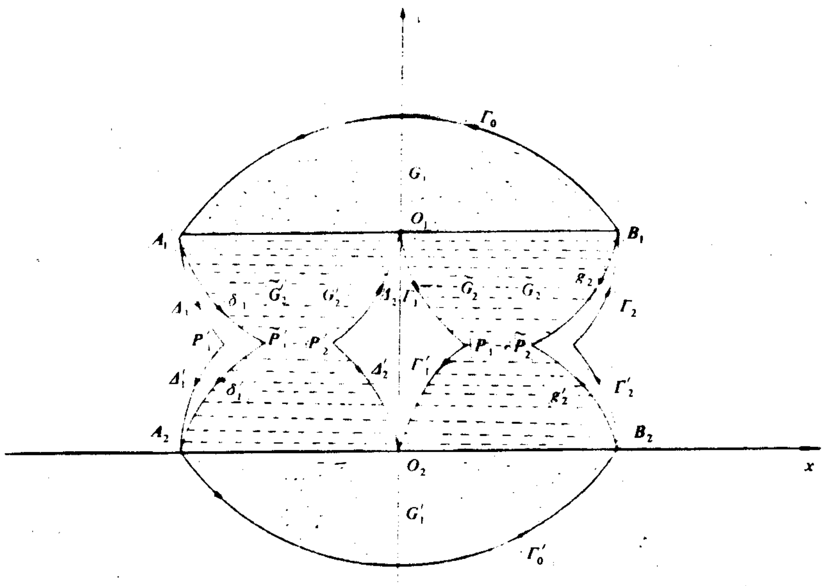
(R_4): $\quad K'>0$ in G_1, and $K'<0$ in G_1'.

Then Problem (ET) has at most one quasi-regular solution in the mixed domain D.

Proof We apply the b,c-energy integral method ($a=0$ in D) and use (++)

First, we assume u_1,u_2: two quasi-regular solutions satisfying equation (+) and boundary conditions (++). Then claim that

$$u=u_1-u_2=0\quad\text{in}\quad D.$$



It is clear now that

$$Lu = K(y)\cdot u_{xx} + u_{yy} + r(x, y)\cdot u = 0, \text{ and}$$

$$u = 0 \quad \text{on Ext}(D).$$

It is enough to show that

$$u = u_1 - u_2 = 0 \quad \text{on Int}(D).$$

Second, investigate

$$0 = J = 2\cdot\iint_D (b\cdot u_x + c\cdot u_y)\cdot Lu\cdot \mathrm{d}x\mathrm{d}y,$$

where

$$(c): \quad \begin{cases} b = x, & c = y-1 \quad \text{in } G_1 \\ b = x, & c = 0 \quad \text{in } G_2 \cup G_2' \\ b = x, & c = y \quad \text{in } G_1'. \end{cases}$$

Then consider the identities

$$2\cdot b\cdot r\cdot u\cdot u_x = (b\cdot r\cdot u^2)_x - (b\cdot r)_x\cdot u^2,$$

$$2\cdot c\cdot r\cdot u\cdot u_y = (c\cdot r\cdot u^2)_y - (c\cdot r)_y\cdot u^2,$$

$$2\cdot b\cdot K\cdot u_x\cdot u_{yy} = (b\cdot K\cdot u_x^2)_x - b_x\cdot K\cdot u_x^2,$$

$$2\cdot b\cdot u_x\cdot u_{yy} = (2\cdot b\cdot u_x\cdot u_y)_y - (b\cdot u_y^2)_x + b_x\cdot u_y^2,$$

$$2\cdot c\cdot X\cdot u_y\cdot u_{xx} = (2\cdot c\cdot K\cdot u_x\cdot u_y)_x - (c\cdot K\cdot u_x^2)_y + (c\cdot K)_y\cdot u_x^2,$$

$$2\cdot c\cdot u_y\cdot u_{yy} = (c\cdot u_y^2)_y - c_y\cdot u_y^2.$$

Then employing above identities and applying Green's theorem we obtain:

$$0 = J = \iint_D [-(b\cdot r)_x\cdot u^2 - (c\cdot r)_y\cdot u^2 - b_x\cdot K\cdot u_x^2 + b_x\cdot u_y^2 + (c\cdot K)_y\cdot u_x^2 - c_y\cdot u_y^2]\mathrm{d}x\mathrm{d}y$$

$$+\oint_{\partial D}[b\cdot r\cdot u^2\cdot v_1 + c\cdot r\cdot u^2\cdot v_2 + b\cdot K\cdot u_x^2\cdot v_1 + 2\cdot b\cdot u_x\cdot u_y\cdot v_2 - b\cdot u_y^2\cdot v_1$$

$$+ 2\cdot c\cdot K\cdot u_x\cdot u_y\cdot v_1 - c\cdot K\cdot u_x^2\cdot v_2 + c\cdot u_y^2\cdot v_2]\cdot \mathrm{d}s,$$

where $v=(v_1, v_2): =(\frac{dy}{ds}, -\frac{dx}{ds})$: outer unit normal vector on ∂D.

Therefore

$$0 = -\iint_D [(b\cdot r)_x + (c\cdot r)_y]\cdot u^2\cdot dxdy$$
$$+\iint_D [(-b_x\cdot K + (c\cdot K)_y)\cdot u_x^2 + (b_x - c_y)\cdot u_y^2]\cdot dxdy + \oint_{\partial D}[(b\cdot v_1 + c\cdot v_2)\cdot r]\cdot u^2\cdot ds$$
$$+\oint_{\partial D}[(b\cdot v_1 - c\cdot v_2)\cdot K\cdot u_x^2 + 2\cdot(b\cdot v_2 + c\cdot K\cdot v_1)\cdot u_x\cdot u_y + (-b\cdot v_1 + c\cdot v_2)\cdot u_y^2]\cdot ds$$
$$= I_1 + I_2 + J_1 + J_3 .$$

Claim that all integrals: I_1, I_2, J_1, and J_3 are non-negative.

First The integrals I_1, I_2 are non-negative if the following two conditions hold in D:

(c_1): $$(b_x + c).r + (b\cdot r_x + c\cdot r_y) \leqq 0$$

(c_2): $$\begin{cases} A: = -b_x\cdot K + (c\cdot K)_y \geqq 0 \\ B: = b_x - c_y \geqq 0. \end{cases}$$

Second The integrals J_1, and J_3 are non- negative if the following conditions hold on ∂D:

(c_3): $(b\cdot v_1)\cdot r > 0$ on $\text{Int}(D)$,

(c_4): $b\cdot v_1 + c\cdot v_2 > 0$ or $\Gamma_0 \cup \Gamma_0'$,

(c_5): $b\cdot v_1 \leqq 0$ on $\text{Int}(D)$.

Justification

Condition (c_3); From [+ +] and (c) we get

$$J_4 = \int_{\text{Int}(D)} [(b\cdot v_1)\cdot r]\cdot u^2\cdot ds$$

Therefore, condition (c_3) holds.

Conditions (c_4) and (c_5):

$$J_3 = \int_{\text{Ext}(D)} Q_1\cdot ds + \int_{\text{Int}(D)} Q_2\cdot ds: = J_3^{(1)} + J_3^{(2)},$$

where

$$Q_1: = Q_1(u_x, u_y): = (b\cdot v_1 - c\cdot v_2)\cdot K\cdot u_x^2 + 2\cdot(b\cdot v_2 + c\cdot K\cdot v_1)\cdot u_x\cdot u_y + (-b\cdot v_1 + c\cdot v_2)\cdot u_y^2,$$
$$Q_2: = Q_1(u_x, u_y): = (b\cdot v_1)\cdot K\cdot u_x^2 + 2\cdot(b\cdot v_2)\cdot u_x\cdot u_y + (-b\cdot v_1)\cdot u_y^2$$

are two quadratic forms with respect to u_x, u_y on $\text{Ext}(D)$ and $\text{Int}(D)$, respectively.

From [+ +] we get

$$u_x = N\cdot v_1, \quad u_y = N\cdot v_2 \text{ on } \text{Ext}(D),$$

where

N: = normalizing factor.

Therefore,

$$Q_1: = (b\cdot v_1 + c\cdot v_2)\cdot(K\cdot v_1^2 + v_2^2)\cdot N^2 .$$

But

$K\cdot v_1^2+v_2^2>0$ on $\Gamma_0\cup\Gamma_0'$ (as $K>0$ in $G_1\cup G_1'$),

$K\cdot v_1^2+v_2^2=0$ on $\mathrm{Ext}(D)\setminus\Gamma_0\cup\Gamma_0'$ (as $\Gamma_2,\Gamma_2',\triangle_1,\triangle_1'$ are characteristics)

Therefore,

$$Q_1=Q_1|_{\mathrm{Ext}(D)}:=Q_1|_{\Gamma_0\cup\Gamma_0'}\geqslant 0$$

if (c_4) holds. Therefore, $J_3^{(1)}\geqq 0$ if (c_4) holds.

Also $J_3^{(2)}\geqq 0$ if

$$Q_2:=Q_2|_{\mathrm{Int}(D)}\geqq 0.$$

But on $\mathrm{Int}(D)$:

$$\begin{vmatrix}(b\cdot v_1)\cdot K & b\cdot v_2\\ b\cdot v_2 & -b\cdot v_1\end{vmatrix}:=-b^2\cdot(K\cdot v_1^2+v_2^2):=0,$$

because

$K\cdot v_1^2+v_2^2=0$ on $\mathrm{Int}(D)$ (as $\Gamma_1,\Gamma_1',\triangle_2,\triangle_2'$ are characteristics)

From (c), therefere,

$$Q_2\geqslant 0 \text{ holds if}$$

$$(b\cdot v_1)\cdot K\geqq 0 \text{ and } -b\cdot v_1\geqq 0 \text{ on } \mathrm{Int}(D),$$

or if condition (c_5) holds (as (as $K<0$ in $G_2\cup G_2'$), and the justification is comp-plete.

Reduction of Conditions $(c_1)-(c_5)$ (by using choices (c)):

Conditions (c_3) and (c_5) are reduced to condition

$(R)_1$: $\qquad r\leqslant 0$ on $\mathrm{Int}(D)$,

because

$$x\cdot v_1\leq 0 \text{ on } \mathrm{Int}(D).$$

Also condition (c_4) is reduced to condition:

$$(R)_2:\quad \begin{cases} x\cdot dy-(y-1)\cdot dx\geqq 0 & \text{on } \Gamma_0,\\ x\cdot dy-y\cdot dx\geqq 0 & \text{on } \Gamma_0'.\end{cases}$$

Besides, condition (c_1) is reduced to condition:

$$(R)_3:\quad \begin{cases} 2\cdot r+x\cdot r_x+(y-1)\cdot r_y\leqq 0 & \text{in } G_1\\ r+x\cdot r_x\leqq 0 & \text{in } G_2\cup G_2'\\ 2\cdot r+x\cdot r_x+y\cdot r_y\leqq 0 & \text{in } G_1'.\end{cases}$$

Finally condition (c_2) is reduced to condition:

$(R)_4$: $\qquad K'>0$ in G_1, $:<0$ in G_1'.

because

$$-b_x\cdot K+(c\cdot K)_y=\begin{cases} -K+(K+(y-1)\cdot K')=(y-1)\cdot K'>0 \text{ in } G_1\\ \qquad\qquad \text{if } K'>0 \text{ in } G_1\\ -K>0 \text{ in } G_2\\ -K+(K+y\cdot K')=y\cdot K'>0 \text{ in } G_1'\\ \qquad\qquad \text{if } K'<0 \text{ in } G_1'\end{cases}$$

and $b_x - c_y = 0$ in $G_1 \cup G_1'$, and $b_x - c_y = 1$ in G_2.

Special case

$$(S): K = \operatorname{sgn}(y\cdot(y-1))\cdot|y|^{\alpha}\cdot|y-1|^{\beta}\cdot h(y) \quad \text{in } D.$$

$$\alpha, \beta > 0, \text{ and}$$

$$h = h(y) > 0 \text{ for all } y,$$

where

$$\operatorname{sgn}(y\cdot(y-1)) := \begin{cases} 1, & y > 1 \\ -1, & 0 < y < 1 \\ 1, & y < 0 \end{cases}$$

and $:= 0$ for $y = 0$ and $y = 1$.

Therefore

$$K(y) = \begin{cases} K_1(y) = y^{\alpha}\cdot(y-1)^{\beta}\cdot h(y) > 0, & y > 1. \\ K_2(y) - y^{\alpha}(1-y)^{\beta} h(y) < 0, & 0 < y < 1 \\ K_3(y) = (-y)^{\alpha}\cdot(1-y)^{\beta}\cdot h(y) > 0, & y < 0 \end{cases}$$

and $:= 0$ for $y = 0$ and $y = 1$.

Corollary

If $K = K(y)$ is of the form (S) in D, if conditions (R_1)—(R_3) of Theorem hold, and if

$$R = R(y; \alpha, \beta) = [\alpha\cdot(y-1) + \beta\cdot y]\cdot h(y) + y(y-1)\cdot h'(y)$$

is such that the following condition

(B): $\quad R > 0$ in G_1, and $R < 0$ in G_1'

holds, then Problem (ET) has at most one quasi-regular solution in the mixed domain $D \subset \mathbf{R}^2$.

Remarks

1) It is clear then on the parabolic lines of degeneracy: $y = 1$ and $y = 0$: $\lim_{y\to1^+} R(y; \alpha, \beta) = \beta\cdot h(1) > 0$, and $\lim_{y\to0^-} R(y; \alpha, \beta) = -\alpha\cdot h(0) < 0$ hold, because $\alpha, \beta > 0$ and $h(y) > 0$ for all y in D.

2) If $r :=$ constant, then conditions (R_1) and (R_3) are replaced by only condition (R_1).

3) If

$$\alpha = \beta = 1, \quad h = 1$$

in (S), then

$$K(y) = \operatorname{sgn}(y\cdot(y-1))\cdot|y|\cdot|y-1| := y\cdot(y-1).$$

and condition (B) in Corollary or condition (R_4) in Theorem is not needed.

The Exterior Frankl Problem

Replace cnaracteristics $\Gamma_2, \Gamma_2', \triangle_1, \triangle_1'$ by smooth non-characteristics: $g_2, g_2', \delta_1, \delta_1'$ so that:

(NC): $H: = K\cdot v_1^2 + v_2^2 > 0$ on $g_2 \cup g_2' \cup \delta_1 \cup \delta_1'$, and

i) g_2 emanating from point B_1 lying inside the characteristic truncated triangle $0_1 P_1 P_2 B_1$ and intersecting Γ_1 at most once. This curve g_2 may conincide with Γ_2 near point B_1,

ii) g_2' emanating from point B_2 lying inside the characteristic truncated triangle $0_2 B_2 P_2 P_1$ and intersecting Γ_1' at most once. This curve g_2' may coincide w with Γ_2' near point B_2,

iii) δ_1 emanating from point A_1 lying inside the characteristic truncated trianegle $A_1 P_1' P_2' O_1$ and intersecting $\triangle_2$ at most once. This curve δ_1 may coincide with $\triangle_1$ near point• A_1, and

iv) δ_1' emanating from point A_2 lying inside the characteristic truncated triangle $A_2 O_2 P_2' P_1'$ and intersecting $\triangle_2'$ at most once. This curve δ_1' may coincide with $\triangle_1'$ near point A_2.

Besides assume boundary conditions

$$(F): \quad \begin{cases} u = \phi_1(s) \text{ on } \Gamma_0, & u = \phi_2(s) \text{ on } \Gamma_0' \\ u = \psi_1(x) \text{ on } g_2, & u = \psi_2(x) \text{ on } g_2' \\ u = \psi_3(x) \text{ on } \delta_1, & u = \psi_4(x) \text{ on } \delta_1' \end{cases}$$

The new mixed domain D' is such that:

$$\partial D' = \text{Ext}(D') \cup \text{Int}(D'),$$

$$\text{Ext}(D') = \Gamma_0 \cup \Gamma_0' \cup \text{Nch}(D'), \quad \text{Int}(D') = \text{Int}(D),$$

$\text{Nch}(D') = g_2 \cup g_2' \cup \delta_1 \cup \delta_1'$: the non-characteristic part of D'.

Resides,

$$D' = G_1 \cup G_1' \cup \widetilde{G}_2 \cup \widetilde{G}_2' \cup (A_1 B_1) \cup (A_2 B_2),$$

where

$$\widetilde{G}_2(\subset G_2): = \{(x, y) \in D', 0 < x < 1, 0 < y < 1\}$$

$$\widetilde{G}_2'(\subset G_2'): = \{(x, y) \in D', -1 < x < 0, 0 < y < 1\}$$

with boundary

$$\partial \widetilde{G}_2: = \Gamma_1 \cup \Gamma_1' \cup g_2 \cup g_2' \cup (B_1 O_1) \cup (O_2 B_2),$$

$$\partial \widetilde{G}_2': = \delta_1 \cup \delta_1' \cup \triangle_2 \cup \triangle_2' \cup (O_1 A_1) \cup (A_2 O_2).$$

The above non-characteristic curves intersect as follows:

$$g_2 \cap g_2' = \widetilde{P}_2, \quad \delta_1 \cap \delta_1' = \widetilde{P}_1'.$$

The Exterior Frankl Problem, or Problem (EF)

Consists in finding a function $u = u(x, y)$ which satisfies equation $(+)$ and boundary conditions (F) in the mixed domain D'.

Then it is clear that a corresponding new uniqueness theorem and a corollary hold in the new domain D' under the same conditions as those of the above proved theorem (and the corollary).

The only difference in statement is that we must change $G_2 \cup G_2'$ with

$\widetilde{G}_2 \cup \widetilde{G}_2$ in (R_3)

$(c)'$: $$(b\cdot v_1)\cdot H \geqq 0 \text{ on } N\mathrm{ch}(D')$$

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Proof If RS is a reduced ring then $R \subseteq RS$ is a reduced ring.

Clearly if R is a reduced ring and S is an ordered semigroup by lemma 1 RS is a reduced ring.

Here it is interesting to note if we relax the condition that S need not be ordered then by above example we cannot always assert that RS to be a reduced ring.

But we pose the following problem.

Problem Can Theorem 2 be true for semi-groups which cannot be ordered?

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