Journal of Mathematical Research and Exposition
Vol. 10, No.4, 1990

A Refinement of Hardy-Riesz's Extended Hilbert Inequality*

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Let $\{a_n\}$ and $\{b_n\}$ be any two sequences of non-negative numbers such that $0<\sum_1^\infty a_n^p<\infty$ and $0<\sum_1^\infty b_n^q<\infty$, where $1/p+1/q=1$ $(p>1)$. Then Hardy-Riesz's extension of the Hilbert inequality can be sharpened to the form

$$\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{a_m b_n}{m+n}<\left(\sum_{n=1}^\infty W_n(q)a_n^p\right)^{1/p}\left(\sum_{n=1}^\infty W_n(p)b_n^q\right)^{1/q}, \qquad (1)$$

where $W_n(x)$ $(x=p$ or $q)$ are the weight-coefficients defined by

$$w_n(x)=\frac{\pi}{\sin(\pi/x)}-\frac{n^{1/x}}{(n+1)(x-1)}. \qquad (2)$$

Note that Hardy-Riesz's extended Hilbert inequality is implied by (1) and (2) since

$$\max\{W_n(q), W_n(p)\}<\frac{\pi}{\sin(\pi/q)}=\frac{\pi}{\sin(\pi/p)}.$$

The particular case $p=q=2$ of (1) gives

$$\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{a_m b_n}{m+n}<\left(\sum_{n=1}^\infty\left(\pi-\frac{\sqrt{n}}{n+1}\right)a_n^2\right)^{1/2}\left(\sum_{n=1}^\infty\left(\pi-\frac{\sqrt{n}}{n+1}\right)b_n^2\right)^{1/2}.$$

In fact, this can be slightly improved to the form

$$\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{a_m b_n}{m+n}<\left(\sum_{n=1}^\infty\left(\pi-\frac{1}{\sqrt{n}}\right)a_n^2\right)^{1/2}\left(\sum_{n=1}^\infty\left(\pi-\frac{1}{\sqrt{n}}\right)b_n^2\right)^{1/2} \qquad (3)$$

The proof of (1) depends esentially on a verification of the following inequality

$$\sum_{m=1}^\infty\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/x}<W_n(x),\quad (x>1).$$

However the case for (3) requires a special treatment. All details will appear elsewhere.

Reference

D. S. Mitrinovic, Analytic Inequalities, 1970. §3.9.36.

* Received May. 11, 1989.