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About 'the Super* Bound of the Limit Circles of the Perturbation System $\dot{x}=y+\varepsilon P_n(y)x$ $\dot{y}=-x+\varepsilon Q_n(x)y$ *

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I. Main Results

We consider the perturbation system

$$\begin{cases} \dot{x}=y+\varepsilon P_n(y)x \\ \dot{y}=-x+\varepsilon Q_n(x)y \end{cases} \qquad (I)_\varepsilon$$

where $P_n(y)=\sum_{i=0}^{n}a_iy^i$, $Q_n(x)=\sum_{i=0}^{n}b_ix^i$, $|a_n|+|b_n|\neq 0$.

Theorem 1 When $0<|\varepsilon|\ll 1$, the system $(I)_\varepsilon$ has $m=[\frac{n}{2}]$ limit circles at the most.

Proof From [1] and [2], we can construct the critical function of $(I)_\varepsilon$

$$\phi_1(A)=\int_0^{2\pi}(x^2P_n(y)+y^2Q_n(x))dt$$

where $x=A\cos t$, $y=A\sin t$. Let $m=[\frac{n}{2}]$, and assume $a_{2m+1}=0$ if $n=2m$, then we have

$$\Phi_1(A)=2\pi A^2\sum_{i=0}^{m}\frac{(2i-1)!!}{(2i+2)!!}(a_{2i}+b_{2i})A^{2i}=2\pi A^2\Phi(A)$$

where
$$\Phi(A)=\sum_{i=0}^{m}\frac{(2i-1)!!}{(2i+2)!!}(a_{2i}+b_{2i})A^{2i}$$

Because $\Phi_1(A)$ has the same positive zero points as $\Phi(A)$, $\Phi(A)$ is the critical function of the system $(I)_\varepsilon$. Obviously, the positive zero points of the critical function $\Phi(A)$ are not more than m, therefore when $0<|\varepsilon|\ll 1$, the system $(I)_\varepsilon$ has $m=[\frac{n}{2}]$ limit circles at the most.

From the critical function of Theorem 1 and some results of [1] and [2], we obtain the following corollaries.

Corollary 1 If $a_{2m}+b_{2m}=0$, and there is a natural number $k<m$ such that $a_{2k}+b_{2k}\neq 0$, $a_{2m}+b_{2m}=\cdots=a_{2k+2}+b_{2k+2}=0$, then $(I)_\varepsilon$ has k limit circles at the most.

Corollary 2 If there is a natural number $k<m$, such that $a_{2k}+b_{2k}\neq 0$, a_{2k+2}

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$+b_{2k+2}=\cdots=a_{2m}+b_{2m}=0$, then, when $(a_{2k}+b_{2k})(a_0+b_0)<0$, $(I)_\varepsilon$ has one limit circle at least

Corollary 3 If $\Phi(A)$ has m positive zero points $0<\beta_1<\beta_2<\cdots<\beta_m$, and $0<|\varepsilon|\ll 1$, then the system $(I)_\varepsilon$ just has m limit circles, and when $(-1)^{m-i}(a_{2m}+b_{2m})\varepsilon<0$ (or >0), the limit cirles of $(I)_\varepsilon$ nearby the closed-curve $x=\sqrt{\beta_i}\cos t$, $y=\sqrt{\beta_i}\sin t$ are stable (or unstable), $i=1,\cdots,m$.

Ⅱ. Some Cases In $(\mathrm{I})_\varepsilon$ If $P_n(y)=0$

Consider the perturbation system

$$\begin{cases} \dot{x}=y \\ \dot{y}=-x-\varepsilon f_n(x)y \end{cases} \qquad (\mathrm{II})_\varepsilon$$

corresponding to Lienard equation $\ddot{x}+f_n(x)\dot{x}+x=0$, $f_n(x)=\sum_{i=0}^{n}a_ix^i$, from Theorem 1 we kown that the system $(\mathrm{II})_\varepsilon$ has $m=[\frac{n}{2}]$ limit circles at the most and the critical function of $(\mathrm{II})_\varepsilon$ is $\Phi(A)$

$$\Phi(A)=-\sum_{i=0}^{m}\frac{(2i-1)!!}{(2i+2)!!}a_{2i}A^{2i}$$

From $\Phi(A)$ we obtain

Theorem 2 Suppose $n=5$ $(0<|\varepsilon|\ll 1)$

(1) If $a_2^2-8a_0a_4>0$, $a_4a_2<0$, the system $(\mathrm{II})_\varepsilon$ just has two limit circles.

(2) If $a_4a_0<0$, the system $(\mathrm{II})_\varepsilon$ has only one limit circle.

(3) If $a_4a_2>0$, the system $(\mathrm{II})_\varepsilon$ has no limit circle.

Theorem 3 Suppose $n=3$ $(0<|\varepsilon|\ll 1)$

(1) If $a_2a_0<0$, then the system $(\mathrm{II})_\varepsilon$ has only one limit circle nearby the closed-curve

$$x=2\sqrt{-\frac{a_0}{a_2}}\cos t, \qquad y=2\sqrt{-\frac{a_0}{a_2}}\sin t\ .$$

and when $a_2\varepsilon<0$ (or >0), the limit circle is stable (or unstable).

(2) If $a_2a_0>0$, then $(\mathrm{II})_\varepsilon$ has no limit circle.

Note When $n=3$, if $a_3=a_1=0, a_2=-1, a_0=1$, then $(\mathrm{II})_\varepsilon$ is the perturbation system $\dot{x}=y$, $\dot{y}=-x+\varepsilon(1-x^2)y$ corresponding to the notable Van Der Pol equation $\ddot{x}+\varepsilon(x^2-1)\dot{x}+x=0$. From Theorem 3 we know that the system has only one limit circle nearby the closed-curve $x=2\cos t$, $y=2\sin t$. It is Stable if $\varepsilon>0$ and unstable if $\varepsilon<0$.

Ⅲ about the Super-bound of the Limit Circles of the Perturbation System $\dot{x}=y-\varepsilon F_{n+1}(x)$ $\dot{y}=-x$.

Under the Lienard transform, from Linenard equation $\ddot{x}+\varepsilon f_n(x)\dot{x}+x=0$, we get the corresponding perturbation system

$$\begin{cases} \dot{x} = y - \varepsilon F_{n+1}(x) \\ \dot{y} = -x \end{cases} \qquad (\text{III})_\varepsilon$$

where $$F_{n+1}(x) = \int_0^x f_n(x)\,dx = \sum_{i=0}^{n} \frac{1}{i+1} a_i x^{i+1},$$

With the same way as the proof of Theorem 1, we can get the critical function of $(\text{III})_\varepsilon$

$$\Phi(A) = \sum_{i=0}^{m} \frac{(2i-1)!!}{(2i+2)!!} a_{2i} A^{2i}, \quad m = [\frac{n}{2}]$$

Thus, the perturbation system $(\text{III})_\varepsilon$ has $m = [\frac{n}{2}]$ limit circles at the most, $0 < |\varepsilon| \ll 1$. From the function $\Phi(A)$, we can easily get the results which are presented in Ref. [3.].

References

[1] 叶彦谦，《极限环论》，上海科技出版社，1984年第2版.

[2] 张锦炎，《常微分方程几何理论与分支问题》，北京大学出版社，1981年.

[3] A. Lino, W, demelo and C. C. push, “Lecture Note in Mathematics 597” p. 335—337.

关于扰动系统 $\dot{x} = y + \varepsilon P_n(y)x$，$\dot{y} = -x + \varepsilon Q_n(x)y$ 的极限环的上界

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摘　　要

本文利用Paincare分支理论，给出了扰动系统

$$\begin{cases} \dot{x} = y + \varepsilon P_n(y)x \\ \dot{y} = -x + \varepsilon Q_n(x)y \end{cases}$$

的判定函数，并由此得到了该系统极限环最大个数等结果．同时还讨论了Lienard 方程 $\ddot{x} + f_n(x)\dot{x} + x = 0$ 对应的扰动系统

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \varepsilon f_n(x)y \end{cases} \quad 或 \quad \begin{cases} \dot{x} = y - \varepsilon F_{n+1}(x) \\ \dot{y} = -x \end{cases}$$

的有关问题，很容易得到了文[3]中的有关结论．