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On the Approximation of Continuous Functions by Pal-Type Interpolation Polynomials*

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Let us denote by

$$-1=x_n<x_{n-1}<\cdots<x_2<x_1=1$$

the n distinct zeros of

$$\pi_n(x)=-n(n-1)\int_{-1}^{x}P_{n-1}(t)\,dt=(1-x^2)P'_{n-1}(x),$$

where $P_{n-1}(x)$ is the Legendre polynomial of degree $n-1$ with the normalization $P_{n-1}(1)=1$. The zeros of the derivative $\pi'_n(x)=-n(n-1)P_{n-1}(x)$ are denoted by λ_k^* $(k=1,2,\cdots,n-1)$. The following relation is valid:

$$-1=x_n<x^*_{n-1}<x_{n-1}<\cdots<x_{k+1}<x^*_k<x_k<\cdots<x_2<x^*_1<x_1=1. \qquad (1)$$

Let $f(x)$ be a continuous function defined in $[-1,1]$, that is, $f(x)\in C[-1,1]$. By the general theory of Pal interpolation (see [1,2]), there exists a unique polynomial $Q_n(f,x)$ of degree at most $2n-1$ for which in the case of nodes (1) the following equations hold:

$$Q_n(f,x_k)=f(x_k),\quad Q'_n(f,x^*_k)=y'_k\quad(k=1,2,\cdots,n), \qquad (2)$$

where $x^*_n=-1$, $\{y'_k\}_{k=1}^n$ are arbitrary real numbers. $Q_n(f,x)$ can be called Pal-type interpolation polynomials and represented in the form

$$Q_n(f,x)=\sum_{k=1}^{n}f(x_k)A_k(x)+\sum_{k=1}^{n-1}y'_kB_k(x)+y'_nC_n(x), \qquad (3)$$

where $A_k(x)$ $(k=1,2,\cdots,n)$, $B_k(x)$ $(k=1,2,\cdots,n-1)$ and $C_n(x)$ are defined as in [3]. In this paper, "$O(1)$" and C will always denote different positive constants independent of x, n and t.

On the convergence of $Q_n(f,x)$ as $n\to\infty$ and the degree of approximation, in 1985 Eneduanya [2] proved:

Theorem E Let $f(x)$ be r-times continuously differentiable on the interval $[-1,1]$ and $r\geqslant 1$, $y'_k=f'(x^*_k)$ $(k=1,2,\cdots,n)$. Then for arbitrary $x\in[-1,1]$ and $n\geqslant 2r+3$,

$$|Q_n(f,x)-f(x)|=O(1)\,\omega(f^{(r)};\frac{1}{n})\cdot n^{-r+\frac{3}{2}}\log n, \qquad (4)$$

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where $\omega(f^{(r)};\cdot)$ is the modulus of continuity of $f^{(r)}(x)$.

Recently, Xie [3] has improved statement (4) and given better estimate for the rate of convergence:

Theorem X Under the conditions of theorem E, then

$$|Q_n(f,x)-f(x)|=O(1)\frac{|\pi_n(x)|}{\sqrt{n}}\omega(f^{(r)};\frac{1}{n})n^{-r+1} \tag{5}$$

holds uniformly in $x\epsilon[-1,1]$, $n>2r+3$.

Notice that (5) implies that if $\pi_n(x)=0$ then $Q_n(f,x)=f(x)$, that is, $Q_n(f,x)$ interpolates $f(x)$ at the zeros of $\pi_n(x)$. In this note, the new upper and lower bounded estimates are derived for the error committed in approximating a continuous function $f(x)$ by Gal-type interpolation polynomials.

Devore [4] proved that, for $f(x)\epsilon C[-1,1]$, there exists a sequence of polynomials $\mu_n(x)=\mu_n(f,x)$ of degree $\leqslant 2n-1$ such that.

$$|f(x)-\mu_n(x)|\leqslant C\omega_2(f;\frac{\sqrt{1-x^2}}{n}),\ (-1\leqslant x\leqslant 1),$$

where $\omega_2(f;\cdot)$ is the modulus of smoothness of order 2 of $f(x)$.

Theorem 1 Let $f(x)\epsilon C[-1,1]$ and let $\mu_n(x)$ be the sequnce of polynomials of Devore's theorem, $y'_k=\mu'_n(x_k^*)$ $(k=1,2,\cdots,n)$. Then, for $Q_n(f,x)$ of (2), (3) and $-1\leqslant x\leqslant 1$, we have

$$|Q_n(f,x)-f(x)|=O(1)\{\omega_2(f;\frac{\sqrt{1-x^2}}{n})+\sqrt{n}|\pi_n(x)|\omega_2(f;\frac{1}{n})\}.$$

Let us suppose that some function $\omega_2(t)$ satisfies the following properties

(i) $\omega_2(t)>0$ for $t>0$, $\omega_2(0)=0$, $\omega_2(T)\geqslant\omega_2(t)$ if $T\geqslant t$, $\omega_2(t)$ is continuous for $t>0$,

(ii) $\dfrac{t^2}{\omega_2(t)}$ is monotone increasing for $t>0$,

(iii) $\lim\limits_{t\to 0^+}\dfrac{t^2}{\omega_2(t)}=0$.

Let us denote by $C(\omega_2)$ the class of all continuous functions $f(x)$ in $[-1,1]$ for which $\omega_2(f;t)\leqslant a(f)\omega_2(t)$, where $a(f)>0$ depends only on $f(x)$, $\omega_2(t)$ is defined by (i), (ii) and (iii).

Theorem 2 If we choose $y'_k=0$ $(k=1,2,\cdots,n)$ and $\omega_2(t)$ generally satisfying (i), (ii) and (iii). Then there exists an $f^*(x)\epsilon C(\omega_2)$ and a sequence $\{n_i\}_{i=1}^{\infty}$ such that for $Q_n(f^*,x)$ of (2)、(3), there holds

$$|Q_n(f^*,0)-f^*(0)|\geqslant Cn\omega_2(\frac{1}{n}),\ (n=n_1,n_2,n_3,\cdots)$$

This lower estimate shows that theorem 1 cannot be significantly improved if one considers all functions $f(x)\epsilon C[-1,1]$.

Theorem 3 Under the conditions of theorem E, then

$$|Q_n(f, x) - f(x)| = O(1)\{\omega_{r+2}(f^{(r)}; \frac{1}{n}) n^{-r+\frac{1}{2}} + |\pi_n(x)| \omega_{r+1}(f^{(r)}; \frac{1}{n}) n^{-r+\frac{1}{2}}\}$$

holds uniformly in $x \in [-1, 1]$, $n > 2r+1$. where $\omega_k(f^{(r)}; \cdot)$ $(k = r+1, r+2)$ is the k-th modulus of continuity of $f^{(r)}(x)$.

References

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