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Finite Geometries and PBIB Designs (II)*

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Let q be a prime power, F_q the finite field with q elements, $U_n(F_q)$ the n-dimensional vector space, and $GL_n(F_q)$ the general group of degree n over F_q. $GL_n(F_q)$ can be viewed as a transformation group of $U_n(F_q)$. $U_n(F_q)$ with $GL_n(F_q)$ as its transformation group is called the n-dimensional linear space, and denoted by $LU_n(F_q)$.

Let
$$K=\begin{bmatrix} 0 & I^{(\nu)} \\ -I^{(\nu)} & 0 \end{bmatrix}$$
be a $2\nu\times 2\nu$ matrix over F_q. The $2\nu\times 2\nu$ matrices T over F_q such that $TKT'=K$ form a group with the matrix multiplication as its composition, where T' denotes the transpose of T. This group is called the symplectic group of degree 2ν over F_q (defined by K), and denoted by $Sp_{2\nu}(F_q)$. $Sp_{2\nu}(F_q)$ can also be viewed as a transformation group of $U_{2\nu}(F_q)$. $U_{2\nu}(F_q)$ with $Sp_{2\nu}(F_q)$ as its transformation group is called the 2ν-dimensional symplectic space or symplectic geometry over F_q, and denoted by $SU_{2\nu}(F_q)$.

Let P be an $m\times 2\nu$ matrix with rank m over F_q. We also use the same symbol P to denote the n-dimensional subspace of $SU_{2\nu}(F_q)$ which is spanned by the rows of the matrix P.

It is known that the vectors orthogonal to P form a $(2\nu-m)$-dimensional subspace, which is called the conjugate subspace of P or the conjugation of P, and denoted by P^*.

Let X_0 be a given $(m,0)$-type subspace of $SU_{2\nu}(F_q)$. Taking as treatments the $(m+1,1)$-type subspaces which include X_0, Shen[1] has constructed a number of BIB designs and PBIB designs, Let Y_0 be a given m-dimensional subspace of $LU_n(F_q)$. Taking as treatments the 1-dimensional subspaces of $LU_n(F_q)$ which are not included in Y_0, Yang and Wei[4] have constructed a number of BIB designs and PBIB designs. Let X_0 denote a given (m,s)-type subspace of $SU_{2\nu}(F_q)$. In the present paper, taking as treatments the 1-dimensional subspaces

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of $SU_{2\nu}(F_q)$ which are orthogonal to but not included in X_0, we construct a number of PBIB designs and evaluate their parameters based on some properties of $SU_{2\nu}(F_q)$.

Throuqhout this paper, X_0 denotes a given (m, s)-type subspace of $SU_{2\nu}(F)$. Let W_1 be the set of 1-dimensional subspaces of $SU_{2\nu}(F_q)$ which are orthogonal to but not included in X_0. Let $U_1, U_2 \in W_1$. If $\dim(U_1 \cup U_2) \cap X_0 = 1$, U_1 and U_2 are said to be the first associates of each other. If $\dim(U_1 \cup U_2) \cap X_0 = 0$ and $U_1 \cup U_2$ is a $(2, 0)$-type subspace (i.e. $U_1 \perp U_2$), U_1 and U_2 are said to be the second associates of each other. If $\dim(U_1 \cup U_2) \cap X_0 = 0$ and $U_1 \cup U_2$ is a $(2, 1)$-type subspaces, U_1 and U_2 are said to be the third associates of each other. Let
$R_i = \{(U_1, U_2) \mid U_1, U_2 \in W_1, U_1$ and U_2 are the i-th associates of each other, $i = 1, 2, 3$.

The main results are the following.

Theorem 1 Let G_0 be the subgroup of $Sp_{2\nu}(F_q)$ consisting of the elements which fix X_0. Then G_0 acts transitively on W_1 as well as transitively on R_i $(i = 1, 2, 3)$.

Theorem 2 Let $2s \leq m \leq \nu + s$. Then an (m, s)-type subspace in $SU_{2\nu}(F_q)$ intersects its conjugate subspace in an $(m-2s, 0)$-type subspace.

Let $2s_1 \leq 2\nu - m$, and W_2 be the set of $(2s_1, s_1)$-type subspaces of $Sp_{2\nu}(F_q)$ which are orthogonal to X_0. For the subgroup G_0 in Theorem 1, we have

Theorem 3 G_0 acts transitively on the set W_2.

Let X_1 be a given $(2s, s)$-type subspace, and G_1 the subgroup of $Sp_{2\nu}(F_q)$ consisting of the elements which fix X_1. We have

Theorem 4 Let $2s' \leq m' \leq \nu + s'$, and $s' \geq s$. Then G_1 acts transitively on the set S_1 of (m', s')-type subspaces which include X_1.

Let $2s_1 \leq m_1 \leq \nu + s_1$, $m_1 \leq 2\nu - 2s_1$, and S_2 be the set of (m_1, s_1)-type subspaces of $Sp_{2\nu}(F_q)$ which are orthogonal to X_1. For the subgroup G_1 in Theorem 4, we have

Theorem 5 G_1 acts transitively on S_2.

Let X_2 be a $(\nu-1, 0)$-type subspece, G_2 the subgroup of $Sp_{2\nu}(F_q)$ consisting of the elements which fix X_2. Then we have

Theorem 6 G_2 acts transitively on the set S_3 of $(t, 0)$-type subspaces which orthogonal to X_2 but not included in X_2, where $2 \leq t \leq \nu$.

Theorem 7 Let $\nu \geq 2$ and $2s \leq m \leq \nu + s$. In the symplectic geometry $SU_{2\nu}(F_q)$, X_0 denotes a given (m, s)-type subspace. Take as treatments the 1-dimensional subspaces which are orthogonal but not included in X_0, and define two treatments to be the first associates of each other if their join intersects X_0 in a 1-dimensional subspace, to be the second associates of each other if their join is a $(2, 0)$-type subspace which intersects X_0 in the 0-dimensional subspace, and to be the third associates of each other if their join is a $(2,1)$-type subspace. Then we obtain an association scheme with three associate classes and with the paremeters

$$u=\frac{q^{2\nu-m}-q^{m-2s}}{q-1}, \qquad n_1=q^{m-2s}-1,$$

$$n_2=\frac{q^{2\nu-m-1}-q^{m+1-2s}}{q-1}, \qquad p_{11}^1=n_1-1, \qquad p_{12}^1=0, \tag{1}$$

$$p_{22}^1=n, \qquad p_{22}^2=\frac{q^{2\nu-m-1}-2q^{m-2s+1}+q^{m-2s}}{q-1}.$$

If $m=2s$, then any two treatments must not be the first associates of each other. In this case. Theorem 7 gives an association scheme with two associate calsses as the following theorem shows.

Theorem 8 Let $\nu\geq 2$ and $1\leq s\leq \nu$. In the symplectic geometry $SU_{2\nu}(F_q)$, a $(2s,s)$-type subspace X_1 is given. Take as treatments the 1-dimensional subspaces which are orthogonal to X_1, and define two treatments to be the first (resp. the second) associates of each other if they are orthogonal (resp. nonorthogonal). Then we obtain an association scheme with two associate classes and with the parameters

$$u=\frac{q^{2\nu-2s}-1}{q-1}, \quad n_1=\frac{q^{2\nu-2s-1}-q}{q-1}, \quad p_{11}^1=\frac{q^{2\nu-2s-2}-2q+1}{q-1}. \tag{2}$$

If $m=\nu-1$ and $s=0$ in Theorem 7, then any two treatments must not be the second associates of each other. In this cese, Theorem 7 also gives an association scheme with two associate classes as the following theorem shows.

Theorem 9 Let $\nu\geq 2$. In the symplectic geometry $SU_{2\nu}(F_q)$, a $(\nu-1,0)$-type subspace X_2 is given. Take as treatments the 1-dimensional subspaces which are orthogonal to but included in X_2, and define two treatments to be the first (resp. the second) associates of each other if they are orthogonal (resp. nonorthogonal). Then we obtain an association scheme with two associate classes and with the parameters

$$u=q^{\nu-1}(q+1), \quad n_1=q^{\nu-1}-1, \quad p_{11}^1=q^{\nu-1}-2. \tag{3}$$

Theorem 10 Adopt the association scheme in Theorem 7. Let $2s_1\leq 2\nu-m$. Take as blocks the $(2s_1,s_1)$-type subspaces of $SU_{2\nu}(F_q)$ which are orthogonal to X_0, and define a treatment to be arranged in a block if the latter includes the former both as subspaces. Then we obtain a PBIB design with three associate classes and with the parameters given in (1) and in the following

$$b=N(2s_1,s_1;\ 2\nu-m,\nu-m+s;2\nu),$$

$$k=\frac{q^{2s_1}-1}{q-1}, \quad r=\frac{bk}{u}, \quad \lambda_1=0,$$

$$\lambda_2=\frac{(q-1)^2N(2s_1,s_1;2\nu-m,\nu-m+s;2\nu)N(2,0;2s_1,s_1,2\nu)}{q^{2n+1-4s}(q^{2(\nu-n+s)}-1)(q^{2(\nu-n+s-1)}-1)},$$

$$\lambda_3=\frac{N(2s_1,s_1;2\nu-m,\nu-m+s;2\nu)N(2,1;2s_1,s_1;2\nu)}{N(2,1;2\nu-m,\nu-m+s;2\nu)}.$$

In particular, we have two PBIB designs each with two associate classes as the next two theorems show.

Theorem 11 Setting $n=2s$ in Theorem 10, we obtain a PBIB design with two associate classes and with parameters given in (2) and in the following

$$b=N(2s_1,s_1;2\nu-2s,\nu-s;2\nu),$$

$$k=\frac{q^{2s_1}-1}{q-1},\quad r=\frac{bk}{u},$$

$$\lambda_1=\frac{(q-1)^2N(2s_1,s_1;2\nu-2s,\nu-s;2\nu)N(2,0;2s_1,s_1;2\nu)}{q(q^{2(\nu-s)}-1)(q^{2(\nu-s-1)}-1)},$$

$$\lambda_2=\frac{N(2s_1,s_1;2\nu-2s,\nu-s;2\nu)N(2,1;2s_1,s_1;2\nu)}{N(2,1;2\nu-2s,\nu-s;2\nu)}.$$

Theorem 12 Setting $m=\nu-1$, $s=0$ and $s_1=1$ in Theorem 10, we obtain a PBIB design with two associate classes and with the parameters given in (3) and in the following

$$u=q^{\nu-1}(q+1),\qquad n_1=q^{\nu-1}-1,\quad p_{11}^1=q^{\nu-1}-2,$$

$$b=q^{2(\nu-1)},\quad k=q+1,\quad r=q^{\nu-1},\quad \lambda_1=0,\quad \lambda_1=1.$$

Theorem 13 Adopt the association scheme in Theorem 7. Take the trentments as blocks, and define a treatment to be arranged in a block if they as subspaces are orthogonal. Then we obtain a PBIB design with three associate classes and with the parameters given in (1) and in the following

$$b=u=\frac{q^{2(\nu-m+s)}-1}{q-1}q^{m-2s},$$

$$r=k=\lambda_1=\frac{q^{2(\nu-m+s)-1}-1}{q-1}q^{m-2s},$$

$$\lambda_2=\lambda_3=\frac{q^{2(\nu-m+s-1)}-1}{q-1}q^{n-2s}.$$

In particular, we have two PBIB designs each with two associate classes as the next two theorems show.

Theorem 14 Setting $m=2s$ in Theorem 13, we obtain a PBIB design with two associate classes and with the parameters given in (2) and in the following

$$b=\frac{q^{2(\nu-s)}-1}{q-1},\qquad r=k=\frac{q^{2(\nu-s)-1}-1}{q-1},$$

$$\lambda_1=\lambda_2=\frac{q^{2(\nu-s-1)}-1}{q-1}.$$

Theorem 15 Setting $m=\nu-1$ and $s=0$ in Theorem 13, we obtain a PBIB design with two associate classes and with the parameters given in (3) and in the following

$$b=q^{v-1}(q+1),\quad r=k=\lambda_1=q^{v-1},\quad \lambda_2=0.$$

Theorem 16 Adopt the association scheme in Theorem 8. Let $2s'\leq m'\leq v+s'$, and $s'\geq s$. Take as blocks the (m',s')-type subspaces which include X_1, and define a treatment to be arranged in a block if the latter includes the former both as subspaces. Then we obtain a PBIB design with two associate classes and with the parameters given in (2) and in the following

$$b=N^T(2s,s;m',s';2v),\qquad r=N^T(2s+1,s;m',s';2v),\quad k=\frac{ru}{b},$$
$$\lambda_1=N^T(2s+2,s;m',s';2v),\quad \lambda_2=N^T(2s+2,s+1;m',s';2v).$$

Theorem 17 Adopt the association scheme in Theorem 8. Let $2s_1\leq m_1\leq v+s_1$, and $m_1\leq 2v-2s$. Take as blocks the (m_1,s_1)-type subspaces which are orthogonal to X_1, and define a treatment to be arranged in a block if the latter include the former both as subspaces. Then we obtain a PBIB design with two associate classes and with the parameters gives in (2) and in the following

$$b=N(m_1,s_1;2(v-s),v-s;2v),\qquad k=\frac{q^{m_1}-1}{q-1},\qquad r=\frac{bk}{u},$$
$$\lambda_1=\frac{N(m_1,s_1;2(v-s),v-s;2v)\,N(2,0;m_1,s_1;2v)}{N(2,0;2(v-s),v-s;2v)}$$
$$\lambda_2=\frac{N(m_1,s_1;2(v-s),v-s;2v)\,N(2,1;m_1,s_1;2v)}{N(2,1;2(v-s),v-s;2v)}$$

Theorem 18 Adopt the association scheme in Theorem 9. Let $2\leq t\leq v$. Take as blocks the $(t,0)$-type subspaces which are orthogonal to and not included in X_2. and define a treatment to be arranged in a block if the latter includes the former both as subspaces. Then we obtain a PBIB design with two associate classes and with the parameters given in (3) and in the following

$$b=\frac{(q+1)\prod_{i=v-t+1}^{v-1}(q^i-1)}{\prod_{i=1}^{(t-1)}(q^i-1)}q^{v-t},\quad k=^{t-1},$$

$$r=\frac{\prod_{i=v-t+1}^{v-1}(q^i-1)}{\prod_{i=1}^{(t-1)}(q^i-1)},\quad \lambda_1=\frac{\prod_{i=v-t+1}^{v-2}(q^i-1)}{\prod_{i=1}^{(t-2)}(q^i-1)},\quad \lambda_2=0.$$

References

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有限几何和PBIB设计(Ⅱ)

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在有限辛几何中，取定一个(m, s)型子空间，并取与这个子空间正交且不含于这个子空间的一维子空间作处理，我们构作了一些结合方案和PBIB设计，并计算了它们的参数.

《数学友谊文集Ⅰ》简介

为纪念徐利治教授从事数学教育四十五周年，由王仁宏教授与周蕴时教授主编的《数学友谊文集Ⅰ》于一九九○年九月在吉林大学出版社正式出版．文章全部用英文书写，采用激光打印，印制精良．

文集收入了徐利治教授的学术活动简介和一批数学专家完成的40篇研究论文，计200余页．简介中概述了徐利治教授的主要学术活动及其在渐近分析、函数逼近、计算方法、组合分析及数学方法论等五个方面的学术贡献．例如，“降维展开与边界型求积公式的构造”、“扩展乘数法”、“激烈振荡函数积分渐近展开与积分法”、“大范围收敛迭代法”及在组合数学中的“Gould-Hsu公式”等卓越工作．简介还给出所引的徐利治教授重要论文的出处．

40篇学术论文涉及到函数逼近论、组合数学、泛函分析、数值方法、微分方程、概率统计、计算机科学与数学基础等领域的最新成果．论文作者遍布长城内外，大河上下，他们之中有年近花甲的知名教授，也有风华正茂的青年数学家．他们中大多数是徐利治教授40多年来直接或间接教授过的弟子，也有个别是与徐利治教授有着学术往来的朋友．

出版这个文集对研究徐利治教授的学术思想及其影响无疑有着重要参考价值．同时对上面所涉及到的数学领域的科学研究也有较高的使用价值．用这样一种形式庆祝一个数学家、教育家的70寿辰实在不属多见．我们希望以后的年月中将会有本文集的第二卷、第三卷问世，以纪念其他数学家的杰出活动．

本文集的对象是研究人员，教学人员，研究生和高年级大学生．　(王天明供稿)