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The Proofs of Several Theorems about the Order Relation*

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Abstract

In this note, we use the fundamental rules of the mathematical logic to prove several theorems concerned with the order relation in decision theory, in order to demonstrate that this kind of method seems simpler, clearer, and more strict, and efficient to decrease mistakes. In the maentime, this process laeds to finding a contradition and a flaw in the previous statements of the theorems.

1. Introduction

The many theorems about the order relation, usualy, are introduced at the beginning of many books which deal with the foundation of decision theory [1][4].

The proofs of these theorems are usually fulfilled in ingeniously argumentative statements [1][4]. Admiring author's shrewd inference, a reader often feels troublesome. The reason is that each sentence is a logic inference, so it needs to be reconsidered carefully from logical relations, and afterwards all sentences should be comprehended togther for obtaining whole impressions being often vague. This process, usually, need much time.

In this paper, we use fundamental rules of mathematical logic to prove these theorems, so that the proofs seem simpler, clearer and more strict and the consequences seem easy to be believed and to decrease logical mistakes.

At the same time, just using these methods, we are leaded to finding a contradition and a flaw in the previous statements of the theorems.

In the next section, we introduce the necessary preliminaries. And then in the third section the proofs about several theorems are given, gotten rid of the contradiction and the flaw.

2. Preliminaries

This section include two parts. One concerns with the concepts of the order

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relation, and another with some formula of the mathematical logic. The detail descriptions can be found in the books [1][2][3][4].

Because of the limitation of the symbols of our typewriter, we will use $\odot$, (Δx), and (x) to denote the ordinary symbols ϵ, $(\forall x)$, and $(\exists x)$ respectively.

1) some definitions of relation.

A binary relation R on a set Y is

P1: reflexive if xRx for each $x \odot Y$.

P2: irreflexive if not xRx for each $X \odot Y$.

P3: symmetric if $xRy \rightarrow yRx$ for each $x, y \odot Y$.

P4: asymmetirc if $xRy \rightarrow$ not yRx for each $x, y \odot Y$.

P5: antisymmetric if $(xRy, yRx) \rightarrow y = x$ for each $x, y \odot Y$.

P6: transitive if $(xRy, yRz) \rightarrow xRz$ for each $x, y, z \odot Y$.

P7: negatively transitive, if (not xRy, not yRz) $\rightarrow$ not xRz for each $x, y, z \odot Y$.

P8: connected if xRy or yRx (possible both) for each $x, y \odot Y$.

P9: weakly connected if $x \neq y \rightarrow (xRy$ or $yRx)$ throughout set Y.

2). some mathematical logic rules

Let R, Q, P, S denote different propositions (statements). Let and, or, $\neg$, $\rightarrow$, and $\leftrightarrow$, denote "and", "or", "not", "implication", and "equivalence" respectively.

Let R=1 if a proposition R is true, and let R=0 if a proposition R is false.

$(\Delta x)\mathrm{P}(x)$ means "for all x, $\mathrm{P}(x)$ hold"

$(\mathrm{L}x)\mathrm{P}(x)$ means "there is an x, $\mathrm{P}(x)$ hold"

The following rules are able to be easily understanded.

F1 P and Q = Q and P; P or Q = Q or P

F2 P or (Q and R) = (P or Q) and (P or R)

F3 P and (Q or R) = (P and Q) or (P and R)

F4 $\mathrm{P} \leftrightarrow \mathrm{Q}$ = (P and Q) or ($\neg$ P and $\neg$ Q)

F5 $\mathrm{P} \rightarrow \mathrm{Q} = \neg$ P or Q $= \neg \mathrm{Q} \rightarrow \neg \mathrm{P}$

F6 $\neg$ (P and Q) $= \neg$ P or $\neg$ Q

F7 $\neg$ (P or Q) $= \neg$ P and $\neg$ Q

F8 $(\mathrm{P} \rightarrow \mathrm{R})$ and $(\mathrm{Q} \rightarrow \mathrm{S})$ = (P and Q) $\rightarrow$ (R and S)

F9 $(\Delta x)(\mathrm{P}(x)$ and $\mathrm{R}(x)) = (\Delta x)\mathrm{P}(x)$ and $(\Delta x)\mathrm{R}(x)$

F10 $(\mathrm{L}x)(\mathrm{P}(x)) = (\mathrm{L}x)\mathrm{P}(x)$ anh $(\mathrm{L}x)\mathrm{R}(x)$

F11 P and 0 = 0; P or 0 = P

F12 P and 1 = P; P or 1 = 1

F13P and $\neg$ P = 0; P or $\neg$ P = 1

F14 $\neg(\neg \mathrm{P}) = \mathrm{P}$

F15 P and P = P; P or P = P

Here we only listed the formulae that will be used, and all they are tautologies being always true for any propositions P, Q, R, S etc.

F16 if $P=1$, P and $Q=0$, then $Q=0$

F17 if $P=1$, P and $Q=1$, then $Q=1$

3. Proofs

In this section we will prove 4 theorems and indicate a contradiction and a flaw which appear in the past proofs; in the mean time, we will provide the improved proofs. And only the original proof of first lemma will be quoted, so that it can be used to compare. The others we give only the numbers of the theorems and the pages located in the original books.

Lemma 1 ([1], page 11)

R is negatively transitive if and only if, for all $x, y, z \odot Y$.

$$xRy \to (xRz \text{ or } zRy) \qquad (1)$$

The original proof:

To prove this suppose first that, in violation of (1), (xRy, not xRz, not zRy). Then, if the P7 condition holds, we get not xRy, which contradicts xRy. Hence the P7 condition implies (1). On the other hand, suppose the P7 condition fails with (not xRy, not yRx, xRz). Then (1) must be false. Hence (1) implies the P7 condition.

The new proof:

Let xRy, yRz, and xRz are represented by R, Q, and S respectively.

$$(\text{not } xRy \text{ and not } yRz) \to \text{not } xRz \qquad \text{(definition P7)}$$
$$= xRz \to \text{not } (\text{not } xRy \text{ and not } yRz) \qquad \text{(F5)}$$
$$= xRz \to xRy \text{ or } yRz \qquad \text{(F6)}$$

(note: Since we use tautologies, it is unnecessary for us to prove sufficiency and necessity respectively).

Theorem 3 ([1] page 15, Theorem 2.3)

Suppose $<$ on X is a strict partial order, being irreflexive and transtive. Then

a) exactly one of $x<y$, $y<x$, $x\approx y$ holds for each $x, y \odot X$.

b) $\approx$ is an equivalence.

c) $x\approx y \leftrightarrow (x<z \leftrightarrow y<z$ and $z<x \leftrightarrow z<y$, for all $z \odot X)$.

d) $(x<y, y\approx z) \to x<z$, and $(x\approx y, y<z) \to x<z$.

e) with $<''$ on $X/\approx$ (the set of equivalence classes of X under $\approx$) defined by

$$a <'' b \leftrightarrow x<y \text{ for some } x \odot a \text{ and } y \odot b$$

$<''$ on $X/\approx$ is a strict partial order.

The definition of $\approx$ is

P12 $\quad x \approx y \leftrightarrow (x\mathrm{E}z \leftrightarrow y\mathrm{E}z$, for all $z \odot x)$

But from the definition P12 and part a) of this theorem, we will induce a contradiction and explain that the proposition c) is also not appropriate; furthermore, we will give a new definition and a proof under new definition.

First, we suppose that part a) can be proved from the definitions, i.e., exactly one of $x<y$, $y<x$, $x \approx y$ can hold for each $x, y \odot X$.

Then, according to the definition P12 and logical formulae,

$$x \approx y \leftrightarrow (\Delta z)(x\mathrm{E}z \leftrightarrow y\mathrm{E}z)$$

$$\leftrightarrow (\Delta z)[(x\mathrm{E}z \text{ and } y\mathrm{E}z) \text{ or } (\text{not } x\mathrm{E}z \text{ and not } y\mathrm{E}z)] \qquad \text{(F4)}$$

For each $z \odot X$, only one of [not $x\mathrm{E}z$ and not $y\mathrm{E}z$], ($x\mathrm{E}z$ and $y\mathrm{E}z$) can be true. No matter we can assume that [not $x\mathrm{E}z$ and not $y\mathrm{E}z$] is true for some $z \odot X$, and let

$$\mathrm{T} = \text{not } x\mathrm{E}z \text{ and not } y\mathrm{E}z$$

$$= \text{not (not } x<z \text{ and not } z<x) \text{ and not (not } y<z \text{ and not } z<y) \qquad \text{(definition)}$$

$$= (x<z \text{ or } z<x) \text{ and } (y<z \text{ or } z<y) \qquad \text{(F7)}$$

furthermore, we assume $x<z$, $z<y$ also true for this z, then

$$\mathrm{T} = (1 \text{ or } 0) \text{ and } (0 \text{ or } 1)$$

$$= 1 \text{ and } 1 \qquad \text{(F12)}$$

$$= 1$$

i.e., this z remain T=true under the assumption, $x<z$ and $z<y$. Hence, this kind of z satisfies the requirement of the definition, and

$$(\Delta z)[1 \text{ or } (x\mathrm{E}z \text{ and } y\mathrm{E}z)] = 1 \leftrightarrow x \approx y$$

But $x<z$ and $z<y \rightarrow x<y$, and it is contrary to $x \approx y$.

As to the proposition c), since there exists some $z \odot X$, which is consistent with the definition of $x \approx y$ and it has the property $x>z$ and $z<y$, then it also is contrary to the right part $x<z \leftrightarrow y<z$ of c); meantime, its inverse is nonsensical.

In fact, all $z \odot X$ are divided into two unintersected subsets according to the definition P12, and it is not enough.

Here we suggest a new definition of $\approx$ as the following,

P13 $\quad x \approx y \leftrightarrow (x<z \leftrightarrow y<z$ and $z<x \leftrightarrow z<y$, for all $z \odot X)$

and prove the theorem 3 under the new definition, naturally, canceling part c) of the theorem.

Proof The strict partial order can be rewrited as

P14 $\quad$ (not $x<x$) and ($x<y$ and $y<z \rightarrow x<z$)

a) $\quad x<y$ and $y<x$

$$\rightarrow x<x=0 \tag{P14}$$

i.e., $x<y$ and $y<x$ can not hold simultaneously.

In P13, if let $z=y$, then

$$x\approx y\rightarrow(x<y\leftrightarrow y<y) \text{ and } (y<x\leftrightarrow y<y)$$
$$=(<y\leftrightarrow 0) \text{ and } (y<x\leftrightarrow 0).$$

So only one of $x<y$, $y<x$, $x\approx y$ can hold.

b) Reflexivity

$$(\Delta z)((x<z\leftrightarrow x<z) \text{ and } (z<x\leftrightarrow z<x))=x\approx x.$$

Transitivity

$$\begin{aligned} & x\approx y \text{ and } y\approx w \\ &=(\Delta z)((x<z\leftrightarrow y<z) \text{ and } (z<x\leftrightarrow z<y)) \text{ and} \\ &\quad (\Delta z)((y<z\leftrightarrow w<z) \text{ and } (z<y\leftrightarrow z<w)) && \text{(P13)} \\ &=(\Delta z)((x<z\leftrightarrow y<z) \text{ and } (y<z\leftrightarrow w<z) \text{ and} \\ &\quad (z<x\leftrightarrow z<y) \text{ and } (z<y\leftrightarrow z<w)) && \text{(F1)} \\ &=(\Delta z)[(x<z\leftrightarrow w<z)\ \ z<x\leftrightarrow z<w)] \\ &=x\approx w \end{aligned}$$

Symmetry

$$\begin{aligned} & x\approx y\leftrightarrow(\Delta z)(x<z\leftrightarrow y<z \text{ and } z<x\leftrightarrow z<y) \\ &\leftrightarrow(\Delta z)(y<z\leftrightarrow x<z \text{ and } z<y\leftrightarrow z<x) \\ &\leftrightarrow y\approx x. \end{aligned}$$

d) Suppose $x<y$ and $y\approx z$ are true

$$x<y \text{ and } z<x \text{ and } y\approx z\rightarrow z<y \text{ and } y\approx z\rightarrow 0 \text{ and } y\approx z=0$$
$$z<x \text{ is false.}$$
$$x<y \text{ and } z\approx x \text{ and } y\approx z=x<y \text{ and } x\approx y\rightarrow x<y \text{ and } 0=0.$$
$$z\approx x \text{ is also false.}$$
$$(x<y,\ y\approx z)\rightarrow x<z$$

The proof of $(x\approx y,\ y<z)\rightarrow x<z$ is similar.

e) Suppose $a, b, c\odot X/\approx$ and $x\odot a$, $y\odot b$, $z\odot c$.

Irreflexivity

$$a<''a=(\Delta x)(x<x)=0 \tag{P14}$$

Transitivity

$$\begin{aligned} & a<''b \text{ and } b<''c \\ &=(\Delta x)(\Delta y)(\Delta z)(x<y) \text{ and } (\Delta x)(\Delta y)(\Delta z)(y<z) \\ &=(\Delta x)(\Delta y)(\Delta z)(x<y \text{ and } y<z) \\ &=(\Delta x)(\Delta y)(\Delta z)(x<z) \\ &=a<''c. \end{aligned}$$

Theorem 5 ([4], page 40, theorem 7)*1

*1 We modified the symbols of the original for the consistence with the prerious statements.

Let $<$ be a total order, being connected and transitive*2. Let D be the set of equivalence classes of X with respect to E. Define the relation $<\cdot$ over D as follows:

$$a, b, c \odot D,\ x \odot a, y \odot b, z \odot c.$$

P16 $x<y$ and not $y<x \leftrightarrow a<\cdot b$

Then $<\cdot$ is a chain order, being weakly connected*2, asymmetric, and transsitive, over D.

We think that the condition P16 of this theorem can be weakened, and as long as

P17 $x<y \leftrightarrow a<\cdot b$

then $<\cdot$ is a chain order. Under the difinition P17, we prove theorem 5 as follows.

Proof Asymmetry.

Suppose $a \neq b$, and $a<\cdot b$ and $b<\cdot a$ are true

$$\begin{aligned}&a<\cdot b \text{ and } b<\cdot a\\ &=(\Delta x)(\Delta y)(x<y) \text{ and } (\Delta x)(\Delta y)(y<x)\\ &=(\Delta x)(\Delta y)(x<y \text{ and } y<x)\\ &=(\Delta x)(\Delta y)(x\mathrm{E}y)\end{aligned}$$

It means x, y belong to same equivalence class, i.e., the elements of a and b are in same equivalence class, hence $a=b$, it is contrary to the premise, so $a<\cdot b$, $b<\cdot a$ can not be true simultaneously.

Weak connectivity.

Suppose $a=b$,

$$\begin{aligned}&a<\cdot b \text{ or } b<\cdot a\\ &=(\Delta x)(\Delta y)(x<y) \text{ or } (\Delta x)(\Delta y)(y<x)\\ &=(\Delta x)(\Delta y)(x<y \text{ or } y<x)\\ &=1.\end{aligned}$$

Transitivity

$$\begin{aligned}&a<\cdot b \text{ and } b<\cdot c\\ &=(\Delta x)(\Delta y)(\Delta z)(x<y) \text{ and } (\Delta x)(\Delta y)(\Delta z)(y<z)\\ &=(\Delta x)(\Delta y)(\Delta z)(x<y \text{ and } y<z)\\ &\rightarrow(\Delta x)(\Delta y)(\Delta z)(x<z)\\ &=a<\cdot c.\end{aligned}$$

Reference

[1] Fishburn, P.C. Utility Theory for Decision Making, John Wiley and Sons, New York, 1970.
[2] Mendelson, E. Introduction to Mathematical Logic, D. Van nostrand co. New York, 1979.
[3] Dalen, D. Von. Logic and Structure, Springer-Verlag, Berlin, 1980.
[4] White, D. J. Fundamentals of Decision Theory, North-Holland pub.co., New York, 1976.

*2 This definition is quoted according to the original.