Journal of Mathematical Research and Exposition
Vol.11 No.2, May, 1991

On Injective Simple Modules Over a Noetherian Commutative Ring*

Pan Shizhong

(Dept. of Applied Math., Dalian Univ. of Tech., Dalian)

It is proved in [1] that an injective simple module over a noetherian commutative ring must be projective. Because every noetherian ring R is a finite direct sum of indecomposable rings, say $R=R_1\oplus\cdots\oplus R_n$, every simple R-module S must be a simple R_i-module for some i and every R-homomorphism $\sigma: M\to S$ must satisfy $\sigma(R_jM)=0\ (j\neq i)$. Thus it is easy to know theorem 2 in [1] is equivalent to the following statement which can be proved much easier.

Theorem 1 There is no injective simple module over an indecomposable noetherian commutative ring R except R is a field.

Proof Suppose S is an injective simple R-module, then there is a maximal ideal M of R such that $S\cong R/M$.

If $M=0$, R is a field. Suppose $M\neq 0$. Because of a.c.c. we have $M=Ru_1+\cdots+Ru_n$ for some $u_i\in M\ (i=1,2,\cdots,n)$.

Denote $N=\bigcap_{i=1}^{n}\operatorname{ann} u_i$.

If $N\neq 0$, take an $x\in N$, $x\neq 0$. Thus $xM=0$, and we have a nonzero homomorphism

$$\begin{cases} R/M\to Rx \\ 1+M\mapsto x \end{cases}$$

Hence there is an exact sequence $0\to R/M\to R$. But R/M is injective and R is indecomposable, thus $R\cong R/M$, R is a simple R-module. It contradicts $M\neq 0$.

If $N=0$, we have a non-zero homomorphism

$$\sigma:\quad \begin{cases} R(u_1,\cdots,u_n)\to R/M \\ (u_1,\cdots,u_n)\mapsto 1+M \end{cases}$$

It can be extended to $\bar{\sigma}: R^n\to R/M$ by the injectivity of R/M. So there are $v_i\in R$ $(i=1,\cdots,n)$ such that

$$(0,\cdots,0,\overset{i}{1},0,\cdots,0)\overset{\bar{\sigma}}{\mapsto} v_i+M\quad (i=1,\cdots,n)$$

Hence we have $\bar{\sigma}(u_1,\cdots,u_n)=\sum_{i=1}^{n}u_iv_i+M=0+M$. It also is a contradiction.

Reference

[1] Guo Shanliang Projectivity and Injectivity of Simple Modules over Commutative Rings, Journal of Mathematical Research and Exposition Vol. 9, No.4, 1989. 501—503.

* Received May, 18, 1990.