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Functions Defined by Sequences*

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Abstract

We construct real valued functions from infinite sequences. We also consider some properties of such functions and null sequences.

1. Introduction

Let Z_+ be the set of all positive integers. Let $\langle a_n \rangle$ be a sequence. Let $t>0$ be a positive number. We define $n(t)$ to be a positive integer depending on t as $n(t)=\min\{m\in Z_+ : |a_u - a_v|<t, \text{ for } u,v\geq m\}$. We now define an integer valued function $f(a_n;t)$ as $f(a_n;t)=n(t)$. We study some properties of functions $f(a_n;t)$ defined by sequences $\langle a_n \rangle$ in connection with sequences $\langle a_n \rangle$. We also study null sequences referring Knopp [2]. We have some theorems and examples of graphs of functions defined by known sequences such as $\langle 1/n \rangle$, $\langle 1+r+r^2+\cdots+r^{n-1}: 0<r<1 \rangle$, $\langle 1/n(\log n) \rangle$, and $\langle 1/(k)^n; k$ is a positive integer greater than $1 \rangle$.

2. Functions defined by Sequences

R denotes the set of the real numbers. $S(R)$ denotes the set of all sequences $\langle a_n \rangle$ over R. We define $CS(R)=\{\langle a_n \rangle \in S(R): \langle a_n \rangle$ is a Cauchy sequence$\}$.

Definition 1 Let $\langle a_n \rangle \in S(R)$, $t>0$. We define $n(t)$ as $n(t)=\min\{m\in Z_+: |a_u - a_v|<t \text{ for } u,v\geq m\}$ and a function $f(a_n;t)$ as $f(a_n:t)=n(t)$.

Definition 2 Let $\langle a_n \rangle \in S(R)$. Let $n\in Z_+$. We define $t(n)$ as $t(n)=\inf\{t\in(0,\infty): |a_n - a_m|<t, \text{ for } m\geq n\}$, where inf T denotes a least upper bound of the set T.

Lemma 1 Let $\langle a_n \rangle \in CS(R)$. Then $f(a_n;t(n))=n$ or $f(a_n;t(n))=n+1$.

Proof Suppose that $f(a_n;t(n))\neq n$. Let $\delta(>0)$ be a sufficiently small positive number. Then we have $f(a_n;t(n)+\delta)=n$, and hence $f(a_n;t(n))=n+1$. This proves the lemma 1.

Theorem 1 (1) Let $\langle a_n \rangle \in S(R)$. Let $\delta>0$. If $f(a_n;t)=0$ $t\in(0,\delta)$, then $\langle a_n \rangle$ is not convergent.

(2) Let $\langle a_n \rangle \in CS(R)$. Then, for any $\delta>0$, $f(a_n;t)$ is a piecewise non-increasing continuous function on (δ,∞).

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Proof Example 5 proves (1). We consider (2). By Lemma 1, either $f(a_n; t(n))$ is equal to n or else is equal to $n+1$. Suppose that $f(a_n; t(n))=n$. Then $f(a_n;t)=n$ for t such that $t(n)\leq t\leq t(n-1)$. This proves that $f(a_n;t)$ on (δ,∞) has at most finitely many discontinuities. Since the sequence is Cauchy, it is clear that $f(a_n;t)$ on (δ,∞) is non-decreasing. This proves the theorem.

$\langle a_n\rangle$ in CS(R) is called a decreasing sequence if $a_{n+2}1<a_n$ for all n. We denote by CS(R; $\downarrow 0$) the set of all decreasing Cauchy sequences $\langle a_n\rangle$ such that $\lim_{n\to\infty} a_n=0$. We prove Theorem 2.

Theorem 2 Let $\langle a_n\rangle\in$ CS(R; $\downarrow 0$) and let $n\in Z_+$. Then $t(n)=a_n$.

Proof Let $n\in Z_+$. We recall that $t(n)=\inf\{t\in(0,\infty): |a_n-a_u|<t \text{ for } u\geq n\}$. Therefore we see that $|a_n-a_u|=a_n-a_u<a_n$ for $u\geq n$. This proves the theorem.

3. $\langle a_n=1+r+r^2+\cdots+r^{n-1};\ 0<r<1\rangle$

In this section we consider the sequence $\langle a_n\rangle$ defined by $a_n=1+r+r^2+\cdots+r^{n-1}$ and $0<r<1$. We often use $a_n=(1-r^n)/(1-r)$.

Computation Let $n\in Z_+$. We compute $t(n)=\inf\{t\in(0,\infty): |a_n-a_u|<t, \text{ for } u\geq n\}$ and find it as $t(n)=r^n/(1-r)$.

Theorem 3 Let $0<r<1$ and $a_n=(1-r^n)/(1-r)$. Then we have the following. (1) $f(a_n;t(n))=n$. (2) $f(a_n;t)=n$ for $t\in(t(n),t(n-1))$. (3) $\int_0^1 f(a_n;t)\,dt=1/(1-r)^2$.

Proof (1) We start with $|a_n-a_m|=|(r^m-r^n)/(1-r)|<r^n/(1-r)$ for all $m\geq n$, which shows that $t(n)=r/(1-r)$. (2) follows from the definition of $t(n)$. $\int_0^1 f(a_n;t)\,dt=\sum_{n=1}^{\infty} n(t(n-1)-t(n))=\sum n(r^{n-1}/(1-r)-r^n/(1-r))=\sum nr^{n-1}=(d/dr(1+r+r^2+\cdots)=d/dr(1/(1-r))=1/(1-r)^2$. This proves the theorem.

4. Examples

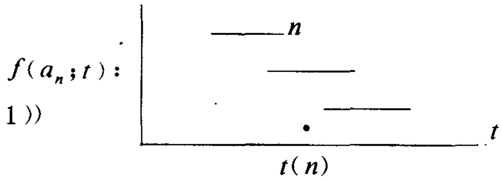
In this section, we have five examples of sequences and their function-graphs.

Example 1 We conside the null sequence $\langle 1/n\rangle=\langle a_n\rangle$.

We obtaine that $t(n)=1/n$ for all n in Z_+. We see that $f(1/n;t(n))=n$, $\int_0^1 f(1/n;t)dt=2(1-1/2)+3(1/3-1/2)+4(1/4-1/3)+\cdots=1+1/2+1/3+1/4+\cdots$ and $\int_{1/n}^1 f(1/n;t)\,dt=1+1/2+1/3+\cdots+1/n$.

Example 2 Letting $a_n=1/k^n$ and $1<k\in Z_+$, consider $\langle a_n\rangle$.

We see that $t(n)=1/k^n$, $f(a_n;t(n))=n$, and $\int_0^1 f(a_n;t)\,dt=1/(1-k)$.



Example 3 Let $a_n=(1/(n+1))(\log(n+1))$

and consider $\langle a_n \rangle$, which is a null sequence. We find that $t(n)=(1/(n+1))\log(n+1)$, $f(a_n; t(n))=n$, and $\int_0^1 f(a_n; t)\mathrm{d}t = 3(1/2\ \log 2 - 1/3\ \log 3) + 4(1/3\ \log 3 - 1/4\ \log 4) + \cdots = 3/2\ \log 2 + 1/3 \log 3 + 1/4 \log 4 + \cdots$, which is divergent.

Example 4 Let $a_n = 1 - 1/n$ and $\langle a_n \rangle$. We know that $\lim_{n\to\infty} a_n = 1$. Let $t = 1/m$. We compute $n(t)$ and find that $n(t)=m$. We see that $f(a_n; n(1/m)) = m$ and $\int_0^1 f(a_n; t)\,\mathrm{d}t = \sum_1^{\infty} n$.

Example 5 Let $\langle (-1)^n \rangle$. We see that $\int_0^1 f(a_n; t)\mathrm{d}t = 0$.

We draw a graph of the function defined by the seqence:

5. The null Sequences

Knopp[2] has a section of the null sequences and we can find some interesting theorems such as [2, p. 60]:

Theorem 4 If $|a|<1$, then besides $\langle a^n \rangle$ even $\langle na^n \rangle$ is a null sequence.

Theorem 5 Let $a_n = \sqrt[n]{n} - 1$. Then $\langle a_n \rangle$ is a null sequence, see [2, P. 62—(5)]. The proof of Theorem 5 is interesting. Let $a \in R$. We first define $\mathrm{CS}(R; a) = [\langle a_n \rangle \in \mathrm{CS}(R): \lim_{n\to\infty} a_n = a\}$ and $\mathrm{CS}(R; a) + b = \{\langle a_n + b \rangle: \langle a_n \rangle \in \mathrm{CS}(R; a)\}$. The following theorem is trivial.

Theorem 6 $\mathrm{CS}(R; a) = \mathrm{CS}(R; 0) + a$.

This theorem indicates that the null sequences $\mathrm{CS}(R; 0)$ are important. We focus attension on $\mathrm{CS}(R; \downarrow 0)$. Let $a_n = \sqrt[n+1]{n+1} - 1$ and $b_n = (1/(n+1))\log(n+1)$. We prove Thegrem 7.

Theorem 7 Let a_n and b_n be defined in the above. Then $\lim_{n\to\infty} a_n/b_n = 1$.

Proof Let $a_n = x - 1$. Then $\lim_{n\to\infty} a_n/b_n = \lim_{x\to\infty}(x-1)/\log x = \lim_{x\to 1} 1/(1/x) = 1$, by L'Hopital's Theorem.

Note Theorem 7 has many meanings and it is important to consider sequences in $\mathrm{CS}(R; \downarrow 0)$. We refer to [2, p. 280] and mention that: $\langle 1/(n\log n) \rangle$, $\langle n^2 \rangle$, $\langle n^3 \rangle$, $\langle 2^n \rangle$, $\langle 3^n \rangle$, $\langle 1/(n!) \rangle$, $\langle 1/(n^n) \rangle$ are such that each congerges (to 0) more rapidly than the preceding.

Table

We have a table for two sequences $a_n = \sqrt[n+1]{n+1} - 1$ and $b_n = (1/(n+1))\log(n+1)$:

1	$b_n =$ 0.150515	$a_n =$ 0.4142135
2	0.1590404	0.4422495
3	0.150515	0.4142135
4	0.139794	0.3797296
5	0.1296918	0.3480061

6	b_n = 0.1207282	a_n = 0.3204692
7	0.1128862	0.2968395
8	0.1060269	0.276518
9	0.1	0.2589254
10	0.0946720	0.2435752
11	0.0899317	0.2300755
12	0.0856879	0.218114
13	0.081662	0.207442
14	0.078406	0.19786
15	0.0752574	0.1892071
16	0.0723793	0.181352
17	0.0697373	0.1741872
18	0.0673028	0.1676234
19	0.0650515	0.1615863
20	0.0629628	0.1560132

References

[1] I. I. Hirschman, Jr., Infinite series, Holt, Rinehart and Winston, New York, 1962.

[2] K. Knopp, Theory and applications of infinite series, Hafner Publication Company, New York, 1971.