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On a Subclass of Starlike Functions of Order α*

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Ⅰ. Introduction

Let A denote the class of functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ that are analytic in the unit disc $D=\{z: |z|<1\}$. Then A is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of D. Denote by S^*, K, $S^*(\alpha)$ and $K(\alpha)$ $(\alpha<1)$ the subclasses of A of starlike, convex, starlike of order α and convex of order α, respectively. A function $f(z)\in A$ is said to be in the class of functions prestarlike of order α $(\alpha<1)$, denoted by $R(\alpha)$, if

$$z/(1-z)^{2(1-\alpha)}*f(z)\in S^*(\alpha),\quad z\in D.$$

Here "$*$" denotes the Hadamard product. In [4], it is shown that $R(\alpha)\subset R(\beta)$, for $\alpha<\beta<1$. It is easily seen that $R(1/2)=S^*(1/2)$, $R(0)=K$.

Let

$$\Phi(a,c;z)=\sum_{n=0}^{\infty}[(a)_n/(c)_n]z^{n+1}\ ,\quad z\in D,\quad c\neq 0,-1,-2,\cdots,$$

$$L(a,c)f=\Phi(a,c)*f(z),\qquad f(z)\in A. \tag{1}$$

Where $(\lambda)_n=\Gamma(n+\lambda)/\Gamma(\lambda)$. It is known [1] that $L(a,c)$ maps A into itself and is

$f(z)$, by

$$D_z^{\lambda} f(z) = \{1/\Gamma(1-\lambda)\}\frac{d}{dz}\int_0^z \{f(\zeta)/(z-\zeta)^{\lambda}\}d\zeta .$$

Where $0\leq\lambda<1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Definition 2 Under the hypothesis of definition 1, the fractional derivative of order $n+\lambda$ is defined by

$$D_z^{\lambda+n} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z)$$

Where $0\leq\lambda<1$, and $n\in N\cup\{0\}$.

Following the lines of Owa and Srivastava [3]. We introduce the linear operator Ω^{λ} defined by

$$\Omega^{\lambda} f = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) = L(2, 2-\lambda) f \qquad 0\leq\lambda<1,\ f(z)\in A. \qquad (4)$$

Further, we define

$$\Omega f = \Omega^1 f = z f'(z) = L(2, 1) f$$

and

$$\Omega^n f = \Omega(\Omega^{n-1} f) = L^n(2, 1) f \qquad n\in N,$$

with, of course, $\Omega^0 f = f(z)$. Then it is easily observed that

$$\Omega^{n+\lambda} f = \Omega^n(\Omega^{\lambda} f) = L^n(2,1) L(2, 2-\lambda) f = \Omega^{\lambda}(\Omega^n f) \qquad (5)$$

for $0\leq\lambda<1$ and $n\in N\cup\{0\}$.

We denote by $A_n(\lambda, a)$ the class of functions $f(z)\in A$ that satisfy the condition

$$\mathrm{Re}\{\Omega^{n+1+\lambda} f/\Omega^{n+\lambda} f\}>a , \quad z\in D \qquad (6)$$

$$\frac{\Omega^{n+\lambda+2}f}{\Omega^{n+\lambda+1}f}=a+(1-a)\frac{1+w(z)}{1-w(z)}+2(1-a)\left[a+(1-a)\frac{1+w(z)}{1-w(z)}\right]^{-1}\frac{zw'(z)}{[1-w(z)]^2}$$
$$z\in D. \tag{8}$$

We claim that $|w(z)|<1$. For otherwise by lemma 1, there exists a z_0, $|z_0|<1$ such that

$$z_0w'(z_0)=kw(z_0). \tag{9}$$

where $|w(z_0)|=1$ and $k\geq 1$. (8) in conjunction with (9), yields

$$\mathrm{Re}\frac{\Omega^{n+\lambda+2}f(z_0)}{\Omega^{n+\lambda+1}f(z_0)}=a\frac{2k(1-a)a}{1+(1-2a)\mathrm{Re}\,w(z_0)\ (1-2a)^2}\leq a-\frac{ka}{2(1-a)}$$

which is a contradiction to our hypothesis. Hence $|w(z)|<1$ and from (7) we conclude that $f(z)\in A_n(\lambda,a)$. This evidently completes the proof theorem 1.

Lemma 2 ([4]) For $a<1$, Let $\Phi(z)\in R(a)$, $g(z)\in S^*(a)$. Let $p(z)$ be analytic and with positive real part in D. then

$$\mathrm{Re}\frac{\Phi(z)*g(z)p(z)}{\Phi(z)*g(z)}>0,\quad z\in D.$$

We extend the results due to Ruscheweyh to the following case.

Lemma 3 For $a\leq\beta<1$, Let $\Phi(z)\in R(a)$, $g(z)\in S^*(\beta)$. Let $p(z)$ be analytic and $\mathrm{Re}\{p(z)\}>\gamma$ in D. then

$$\mathrm{Re}\frac{\Phi(z)*g(z)p(z)}{\Phi(z)*g(z)}>\gamma\quad 0\leq\gamma<1,\quad z\in D.$$

Proof Let $G(z)=p(z)-\gamma$. then $\mathrm{Re}\{G(z)\}>0$, We have

$$\Phi(z)*g(z)G(z)=\Phi(z)*g(z)p(z)-\gamma[\Phi(z)*g(z)]. \tag{10}$$

Note that $g(z)\in S^*(\beta)\subset S^*(a)$, divided both sides of (10) by $\Phi(z)*g(z)$ and the result follows immediately from lemma 2.

Theorem 2 Let $\Phi(z)\in R(a)$, $0\leq a\leq\beta<1$, $0\leq\lambda<1$ and $n\in N\cup\{0\}$, if $f(z)\in A_n(\lambda,\beta)$, then $\Phi(z)*f(z)\in A_n(\lambda,\beta)$.

Proof Let $f(z)\in A_n(\lambda,\beta)$, then $\Omega^{n+\lambda}f\in S^*(\beta)$. Note that

$$\mathrm{Re}\frac{\Omega^{n+\lambda+1}(\Phi*f)}{\Omega^{n+\lambda}(\Phi*f)}=\mathrm{Re}\frac{\Phi*\Omega^{n+\lambda+1}f}{\Phi*\Omega^{n+\lambda}f} \tag{11}$$

since $\Phi(z)\in R(a)$, $p(z)=\Omega^{n+\lambda+1}f/\Omega^{n+\lambda}f$ be analytic in D and $\mathrm{Re}\{p(z)\}>\beta$, lemma 3 in conjunction with (11), yields $\Phi(z)*f(z)\in A_n(\lambda,\beta)$, The proof of Theorem 2 is, therefore, complete.

Corollary 1 For $0\leq\beta<1$, if $\Phi(z)\in K$, $f(z)\in S^*(\beta)$, then $\Phi(z)*f(z)\in S^*(\beta)$. This is duo to Wu Zhuoren [6].

Corollary 2 For $0\leq\beta<1$, if $\Phi(z)\in K$, $f(z)\in K(\beta)$, then $\Phi(z)*f(z)\in K(\beta)$. In particular, taking $\beta=0$, this is well-known Polya-Schoenberg conjucture which has been proved.

Corollary 3 Let $\Phi(z)\in S^*(1/2)$, $f(z)\in S^*(1/2)$, then $\Phi(z)*f(z)\in S^*(1/2)$. This is duo to S. Ruscheweyh and T. S. Small [7].

Corollary 4 For $0 \leq a < 1$, let $f(z) \in A_n(\lambda, a)$, then

$$F(z) = [(c+1)/z^c] \int_0^z t^{c-1} f(t)\, dt \in A_n(\lambda, a),$$

where c is a complex number and $\mathrm{Re}\, c \geq 0$.

Theorem 3 $A_n(\lambda, a) \subset A_n(u, a)$ holds for all $0 \leq u < \lambda < 1$, $u/2 \leq a < 1$ and $n \in N \cup \{0\}$.

Proof Let $f(z) \in A_n(\lambda, a)$. Note that

$$\Omega^{n+u} f = L^n(2,1) L(2, 2-u) f = L^n(2,1) L(2-\lambda, 2-u) L(2, 2-\lambda) f = L(2-\lambda, 2-u) \Omega^{n+\lambda} f$$

since $0 \leq u < \lambda < 1$, $u/2 \leq a < 1$. we observe that

$$L(2-\lambda, 2-u)[z/(1-z)^{2-u}] = z/(1-z)^{2-\lambda} \in S^*(\lambda/2) \subset S^*(u/2).$$

It follows from (1) that $\Phi(2-\lambda, 2-u) \in R(u/2) \subset R(a)$. By using Theorem 2, we have $\Phi(2-\lambda, 2-u) * f(z) \in A_n(\lambda, a)$. This implies $\Omega^{n+u} f = \Omega^{n+\lambda}[\Phi(2-\lambda, 2-u) * f(z)] \in S^*(a)$, Hence, we have $f(z) \in A_n(u, a)$. That is $A_n(\lambda, a) \subset A_n(u, a)$.

Corollary 5 If $0 \leq \lambda < 1$, $0 \leq a < 1$ and $n \in N \cup \{0\}$, then $A_n(\lambda, a)$ is a subclass of starlike functions of order a. Hence a function $f(z)$ belonging to $A_n(\lambda, a)$ is univalent.

Since $f(z) \in A_n(\lambda, a)$ if and only if $\Omega^{n+\lambda} f \in S^*(a)$. From this we can eatablish.

Theorem 4 If $0 \leq \lambda < 1$, $0 \leq a < 1$ and $n \in N \cup \{0\}$, then $f(z) \in A_n(\lambda, a)$ if and only if

$$f(z) = L^n(1,2) L(2-\lambda, 2) \{z \cdot \exp[2(1-a) \int_0^{2\pi} \log(1 - e^{-it} \cdot z)^{-1} d\mu(t)]\}$$

where $\mu(t)$ is a probability measure on $[0, 2\pi]$.

Theorem 5 For $0 \leq \lambda < 1$, $0 \leq a < 1$ and $n \in N \cup \{0\}$, let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, if $f(z) \in A_n(\lambda, a)$, then

$$|a_k| \leq [\Gamma(k+1-\lambda)/\Gamma(k+1)^2 k^{n-1} \Gamma(2-\lambda)] \prod_{m=2}^{k} (m-2a), \quad k = 2, 3, \cdots$$

The result is sharp and the extreme function is $f(z) = L^n(1,2) L(2-\lambda, 2)[z/(1-z)^{2(1-a)}]$.

It is well known that if $f(z) \in S^*(a)$ $(0 \leq a < 1)$, then

$$|f'(z)| \leq \frac{d}{dr}[r/(1-r)^{2(1-a)}] = L(2,1) \Phi[2(1-a), 1; r]/r \quad r = |z|, \tag{12}$$

$$|f(z)| \leq r/(1-r)^{2(1-a)} = \Phi[2(1-a), 1; r] \quad r = |z|, \tag{13}$$

with the aid of this fact, we can prove the following result.

Theorem 6 Let $0 \leq \lambda < 1$, $0 \leq a < 1$ and $n \in N \cup \{0\}$, If $f(z) \in A_n(\lambda, a)$, then

$$|f(z)| \leq L^n(1,2) L(2-\lambda, 2) \Phi[2(1-a), 1; r]. \tag{14}$$

$$|f'(z)| \leq L^n(1,2) L(2-\lambda, 1) \Phi[2(1-a), 1; r]/r. \tag{15}$$

In addition, we have

$$|f(z)| \geq -L^n(1,2) L(2-\lambda, 2) \Phi[2(1-a), 1; -r], \quad n \geq 1. \tag{16}$$

$$|f'(z)| \geq -L^n(1,2) L(2-\lambda, 1) \Phi[2(1-a), 1; -r]/r \quad n \geq 2. \tag{17}$$

where $r = |z|$. All results are sharp.

Proof It is clear from (12) and (13) that (14) and (15) are true for $\lambda=0$ and $n=0$. Suppose now $f(z)\in A_0(\lambda,a)$ and Fix $0<\lambda<1$. Then $g(z)=L(2,2-\lambda)f\in S^*(a)$, or write

$$f(z)=L(2-\lambda,2)g.$$

Further, we obtain $$zf'(z)=L(2-\lambda,2)[zg'(z)].$$

Note that $2>2-\lambda>0$ $(0<\lambda<1)$. By using (2) and (12), yields

$$\begin{aligned}|zf'(z)|&=\left|\int_0^1 u^{-1}[zug'(zu)]\mathrm{d}\mu(2-\lambda,\lambda)(u)\right|\\ &\leq\int_0^1 u^{-1}L(2,1)\Phi[2(1-a),1;ur]\mathrm{d}\mu(2-\lambda,\lambda)(u)\\ &=L(2-\lambda,1)\Phi[2(1-a),1;r]. \quad r=|z|.\end{aligned}$$

This is

$$|f'(z)|\leq L(2-\lambda,1)\Phi[2(1-a),1;r]/r, \quad r=|z|. \tag{18}$$

Since $f(z)=L(1,2)zf'(z)$. the same method lead to the upper bound of $|f(z)|$ from (18). We note that $f(z)\in A_{n+1}(\lambda,a)$ if and only if $zf'(z)\in A_n(\lambda,a)$. From this it follows that (14) and (15) hold.

Next, let $f(z)\in A_1(\lambda,a)$, then $\Omega^\lambda f\in K(a)$. We can write

$$f(z)=L(2-\lambda,2)h(z), \quad h(z)\in K(a). \tag{19}$$

It is shown [11] that if $h(z)\in K(a)$ $(0\leq a<1)$, then

$$\mathrm{Re}\{h(z)/z\}\geq -L(1,2)\Phi[2(1-a),1;-r]/r, \quad r=|z|. \tag{20}$$

This implies that (16) holds for $n=1$ and $\lambda=0$. Let $0<\lambda<1$, and making use of (2), (19) and (20), we have

$$\begin{aligned}|f(z)|&\geq r\mathrm{Re}\{f(z)/z\}\\ &\geq\int_0^1 u^{-1}\{-L(1,2)\Phi[2(1-a),1;-ur]\}\mathrm{d}\mu(2-\lambda,\lambda)(u)\\ &=-L(1,2)L(2-\lambda,2)\Phi[2(1-a),1;-r]. \quad r=|z|.\end{aligned} \tag{21}$$

Since $f(z)\in A_{n+1}(\lambda,a)$ if and only if $zf'(z)\in A_n(\lambda,a)$. The common techniques lead to (16) and (17). By considering the function $f(z)=L^n(1,2)L(2,-\lambda,2)\Phi[2(1-a),1;z]$, one can show that all results are sharp. Thus we complete the proof of Theorem 6.

The problem of studying the convex hulls and the extreme points of various families of univalent functions was initiated by three of the authors in [9].We shall use the same notation with the exception that HF shall now denotes the closed-convex hulls of a family of functions F. EHF shal denotes the set of extreme points of HF.

A point $f\in P$ is called a support point of P if there exists a continuous linear functional J on A such that

$$\mathrm{Re}\{J(f)\}=\max\{\mathrm{Re}\,J(g):\text{ for all } g\in P\}$$

and $\mathrm{Re}\{J\}$ is non-constant on P. The set of support points of P will be denoted by $SUPP\{P\}$.

Lemma 4 [8] Let $0 \leq a < 1$, $\mathscr{P}$ the set of probability measure on $|x| = 1$, then

$$HS^*(a) = \{\int_{|x|=1} [z/(1 - xz)^{2(1-a)}] \mathrm{d}\mu(x) : \mu(x) \in \mathscr{P}\},$$
$$EHS^*(a) = \{z/(1 - xz)^{2(1-a)} : |x| = 1\}.$$

Lemma 5 Let X, Y be linear topological space. J is a linear topological mapping from X to Y. If $B \subset X$, then

(i) $HJ(B) = J(HB)$. (ii) $EHJ(B) = J(EHB)$.

With the aid of lemma 4 and lemma 5, we can prove

Theorem 7 Let $0 \leq \lambda < 1$, $0 \leq a < 1$ and $n \in N \cup \{0\}$, $\mathscr{P}$ the set of probability measure on $|x| = 1$. then

$$HA_n(\lambda, a) = \{\int_{|x|=1} L^n(1,2) L(2-\lambda, 2) [z/(1 - xz)^{2(1-a)}] \mathrm{d}\mu(x); \mu \in \mathscr{P}\}.$$
$$EHA_n(\lambda, a) = \{L^n(1,2) L(2-\lambda, 2) [z/(1 - xz)^{2(1-a)}] : |x| = 1\}.$$

Proof We frist prove Theorem 7 is true for $n = 0$. Define linear operator $L(2-\lambda, 2): A \mapsto A$. From introduction, we know that $L(2-\lambda, 2)$ is a linear homeomorphism from A to A. Note that

$$L(2-\lambda, 2) S^*(a) = A_n(\lambda, a),$$
$$L(2-\lambda, 2) \int_{|x|=1} [z/(1 - xz)^{2(1-a)}] \mathrm{d}\mu(x) = \int_{|x|=1} L(2-\lambda, 2) [z/(1 - xz)^{2(1-a)}] \mathrm{d}\mu(x).$$

The results follows immediately from lemma 4 and lemma 5.

For $n \geq 1$, Note that $L(1,2) A_n(\lambda, a) = A_{n+1}(\lambda, a)$. Making use of above result, the conclusion of Theorem 7 follows at once.

Lemma 6 ([9]) Let $\{b_n\}$ be a sequence of complex numbers such that $\lim\limits_{n \to \infty} \sup |b_n|^{1/n} < 1$ and set $J(f) = \sum_{n=0}^{\infty} a_n b_n$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f \in A$. Then J is a continuous linear functional on A. Conversely, any continuous linear functional on A is given by such a sequence $\{b_n\}$.

Lemma 7 ([10]) If $0 \leq a < 1$, then

$$SUPPS^*(a) = EHS^*(a) = \{z/(1 - xz)^{2(1-a)}, |x| = 1\}.$$

Theorem 8 If $0 \leq \lambda < 1$, $0 \leq a < 1$ and $n \in N \cup \{0\}$, then

$$SUPPA_n(\lambda, a) = EHA_n(\lambda, a) = \{L^n(1,2) L(2-\lambda, 2) [z/(1 - xz)^{2(1-a)}], |x| = 1\}.$$

Proof From Theorem 6, we know that $A_n(\lambda, a)$ is locally uniformly bounded (in fact, it is compact), We first prove that Theorem 8 is true for the case $n = 0$.

Take any $f_0(z) = L(2-\lambda, 2) [z/(1 - x_0 z)^{2(1-a)}] = z + (2-\lambda)(1-a) x_0 z^2 + \cdots$ belonging to $EHA_n(\lambda, a)$. Let $J(g) = (1/2) \overline{x}_0 g''(0)$, $g \in A$, then functional J be continuous and linear on A. By Theorem 5, we have

$$\mathrm{Re}\, J(f_0) = (2-\lambda)(1-a) = \max\{\mathrm{Re}\, J(g) ; g \in EHA_0(\lambda, a)\}$$

$$= \max\{\mathrm{Re}\, J(g) ; g \in A_0(\lambda, a)\}$$

Hence $f_0(z) \in SUPPA_0(\lambda, a)$, this implies that $EHA_0(\lambda, a) \subset SUPPA_0(\lambda, a)$.

Conversely, Let the functional J be continuous and linear on A and not constant on $A_0(\lambda, a)$, and let J be given be the sequence $\{b_n\}$ described by Lemma 6. Define the sequence $\{b'_n\}$ by $b'_0 = b_0$ and $b'_n = b_n \Gamma(n+2-\lambda)/\Gamma(2-\lambda)\Gamma(n+2)$ for $n = 1, 2, \cdots$, and thus $\limsup_{n\to\infty} |b'_n|^{1/n} = \limsup_{n\to\infty} |b_n|^{1/n} < 1$. Therefore, the functional J', defined by $J'(f) = \sum_{n=0}^{\infty} b'_n a_n$ when $f(z) \in A$ and $J(f) = \sum_{n=0}^{\infty} a_n z^n$, is continuous and linear on A. Also, J' is not constant on $S^*(a)$.

The mapping from $A_0(\lambda, a)$ to $S^*(a)$ defined by $g(z) = L(2, 2-\lambda) f(z)$ is a linear homeomorphism and for this mapping $J'(g) = J(f)$. Therefore, take any $f_0 \in SUPPA_0(\lambda, a)$, then $g_0(z) = L(2, 2-\lambda) f_0(z) \in S^*(a)$, and

$$\begin{aligned} \mathrm{Re}\, J'(g_0) &= \mathrm{Re}\, J(f_0) \\ &= \max\{\mathrm{Re}\, J(f) : f \in A_0(\lambda, a)\} \\ &= \max\{\mathrm{Re}\, J'(g) ; g \in S^*(a)\} \end{aligned}$$

which implies that $g_0(z) \in SUPPS^*(a)$, by lemma 7, we obtain

$$f_0(z) = L(2-\lambda, 2) g_0(z) = L(2-\lambda, 2)\,[z/(1-x_0 z)^{2(1-a)}] \in EHA_0(\lambda, a)$$

that is, $SUPPA_0(\lambda, a) \subset EHA_0(\lambda, a)$.

For $n \geq 1$, note that $L(2,1) A_{n+1}(\lambda, a) = A_n(\lambda, a)$. Applying the above results, we can lead to the assertion of Theorem 8.

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关于 α 阶星像函数类的一个子类

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设$f(z)=z+\cdots$在单位圆$|z|<1$内解析．给定$\lambda(0\leq\lambda<1)$，我们定义线性算子Ω^{λ}

$$\Omega^{\lambda}f=\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z);$$

和

$$\Omega^{n+\lambda}f=\Omega^{n}(\Omega^{\lambda}f),\quad n\in N\cup\{0\}.$$

其中$D_z^{\lambda}f(z)$表示$f(z)$的分数阶导数且$\Omega^{1}f(z)=zf'(z)$．用$A_n(\lambda,\alpha)$表示满足

$$\mathrm{Re}\frac{\Omega^{n+1+\lambda}f}{\Omega^{n+\lambda}f}>\alpha,\quad 0\leq\alpha<1$$

的所有函数$f(z)$之集．在本文，对$0\leq\lambda<1$，$0\leq\alpha<1$和$n\in N\cup\{0\}$．我们证明了$A_n(\lambda,\alpha)$是α阶星像函数类的一个子类．同时，我们也得到了$A_n(\lambda,\alpha)$的系数不等式、偏差定理、极值点和支撑点．