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Littlewood-Paley Operators on the Space of Functions of Weighted Bounded Mean Oscillation*

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Abstract

Littlewood-Paley operators, g-function, s-function and g_λ^*-function ($\lambda>3$), considered as operators on the space of functions of weighted bounded mean oscillation BMO_w ($(BMO)_w$), are "bounded operators". Exactly, we proved that if f $BMO_w((BMO)_w)$ and $|\{x: Tf(x)\neq\infty\}|>0$, then Tf is also in $BMO_w((BMO)_w)$ and there is a constant C independent of f such that $\|Tf\|_{BMO_w}\leqslant C\|f\|_{BMO_w}$ ($\|Tf\|_{BMO_w}\leqslant C\|f\|_{(BMO)_w}$), where T is one of those Littlewood-Paley operators.

§1 Introduction

In [1], B. Muckenhoupt and R.L. Wheeden introduced the space of functions of weighted bounded mean oscillation.

Let $f(x)$ and $W(x)$ be locally integrable in R^n and $w\geqslant 0$, we say that f is of bounded mean oscillation with weight w if there is a constant C such that

$$\int_Q|f(x)-f_Q|dx\leqslant C\int_Q w(x)dx, \text{ where } f_Q=1/|Q|\int_Q f(x)dx \tag{1.1}$$

for all n-dimensional "cubes" Q whose sides are parallel to the coordinate axes.

We write $BMO_w=\{f: f\in L^1_{loc}(R^n)$ and f satisfies (1.1)$\}$. The smallest constant C which satisfies (1.1) is called the BMO_w norm of f and is denoted by $\|f\|_{w,*}$.

Another weighted definition of this class is also given as follows:

$$\int_Q|f(x)-(f)_{Q,w}|w(x)dx\leqslant C\int_Q w(x)dx, \tag{1.2}$$

where $(f)_{Q,w}=\int_Q f(x)w(x)dx/\int_Q w(x)dx$ for all Q.

And we write $(BMO)_w=\{f: f$ $L^1_{loc}(R^n)$ and f satisfies (1.2)$\}$. The smallest constant C which satisfies (1.2) is called the $(BMO)_w$ norm of f and is denoted by $\|f\|_{*,w}$.

Wang [7], Yao [8], Han [9] and Kurtz [10] have considered the Littlewood-Paley operators on BMO. Using a similar way of proof, they obtained a lot of

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good results.

Here, we deal with the weighted cases and get the following conclusions.

Theorem 1 Let $w \in A_1$ and $f \in \mathrm{BMO}_w(\mathrm{R}^n)$. Suppose $\lambda > 3$, then either $g(f)(x)$ $(s(f)(x),\ g_\lambda^*(f)(x)) = \infty$ a.e., or $g(f)(x)\ (s(f)(x),\ g_\lambda^*(f)(x)) < \infty$ a.e. and there is a constant C independent of f and x such that

$$\|g(f)\|_{w,*} \leqslant C\|f\|_{w,*} (\|s(f)\|_{w,*} \leqslant C\|f\|_{w,*},\ \|g_\lambda^*(f)\|_{w,*} \leqslant C\|f\|_{w,*}).$$

Theorem 2 Let w A_∞ and f $(\mathrm{BMO})_w$ Suppose >1, then either $g(f)(x)$ $(s(f)(x),\ g_\lambda^*(f)(x)) = \infty$ a.e., or $g(f)(x)\ (s(f)(x),\ g_\lambda^*(f)(x)) < \infty$ a.e. and there is a constant C independent of f and x such that

$$\|g(f)\|_{*,w} (\|s(f)\|_{*,w}, \|g_\lambda^*(f)\|_{*,w}) < C\|f\|_{*,w}.$$

We assume all sets and functions are measurable, using χ_E and $|E|$ to denote the characteristic function and Lebesgue measure of the measurable set E and dQ the cube with the same centre as Q and its edge length being d times as long as that of cube Q.

We also use C to denote constants which may change from line to line and independent of f and x.

§ 2 The proof of Theorem 1

Let $x \in \mathrm{R}^n$, $y > 0$, $f(x, y)$ be the Poisson integral of f, and

$$g(f)(x) = \left(\int_0^\infty y\,|\mathrm{grad}(f(x,y))|^2 \mathrm{d}y\right)^{1/2},$$

$$s(f)(x) = \left(\iint_{\Gamma(x)} y^{1-n} |\mathrm{grad}(f(z,y))|^2 \mathrm{d}z\mathrm{d}y\right)^{1/2},$$

where $\Gamma(x) = \{(z,y)\ \ R_+^{n+1}: |z-x| < y\}$ and

$$g_\lambda^*(f)(x) = \left(\iint_{R_+^{n+1}} (y/(y+|z-x|))^{\lambda n} y^{1-n} |\mathrm{grad}(f(z,y))|^2 \mathrm{d}z\mathrm{d}y\right)^{1/2},$$ where $\lambda > 1$.

The proof of Theorem 1 is based on the following Lemma and Corollary.

Lemma Let $w \in A_p (1 < p < \infty)$, $f \in \mathrm{BMO}_w$ and Q be a cube centred at x_0 and having edge length r. Then there is a constant C depending only on n and p such that for any arbitrary positive $y > 0$,

$$\int_{\mathrm{R}^n} [|f(x) - f_Q| / (y^{np} + |x - x_0|^{np})] \mathrm{d}x$$
$$\leqslant C y^{n-np} (1 + |\ln(y/r)|) \|f\|_{w,*} \max\{w(Q)/|Q|,\ w(\tfrac{y}{r}Q)/|\tfrac{y}{r}Q|\}.$$

Corollary Let $w \in A_1$, $f \in \mathrm{BMO}_w$, $\varepsilon > 0$ and Q as in the Lemma. There is a constant C depending only on n and ε such that

$$\int_{\mathrm{R}} \qquad r^{n+\varepsilon} + |x - x_0|^{n+\varepsilon}) |\mathrm{d}x \leqslant \mathrm{C} r^{-\varepsilon} \|f\|_{w,*} w(Q) / |Q|.$$

Certainly,

$$\int_{\mathrm{R}^n} [|f(x) - f_Q| / (r + |x - x_0|)^{n+\varepsilon}] \mathrm{d}x \leqslant \mathrm{C} r^{-\varepsilon} \|f\|_{w,*} w(Q) / |Q|,$$

where $w(Q) = \int_Q w(x)\mathrm{d}x$.

For $w\in A_1$, by the property of Muckenhoupt class, we also have $w\in A_{(n+\varepsilon)/n}$. Taking $p=(n+\varepsilon)/n$ in the Lemma, the Corollary is immediate.

In particular, if we take Q_0 to be the unit cube, $\varepsilon=1$, then

$$\int_{R^n}[|f(x)-f_{Q_0}|/(1+|x|)^{n+1}]dx\leq C\|f\|_{w,*}w(Q_0).$$

It follows that $\int_{R^n}[|f(x)|/(1+|x|)^{n+1}]dx<\infty$, if $f\in BMO_w$. As we know, this condition is equivalent to the finiteness of the Poisson integral $f(x,y)$, $y>0$, of f, and it is possible to consider Littlewood-Paley operators on BMO_w.

Proof (Lemma) Firstly, we prove the Lemma on the special case: $y=r$.

Let $Q(k)=2^kQ, Q(0)=Q$, $(k=1,2,\cdots)$, then

$$|f_{Q(k)}-f_{Q(k-1)}|\leq(1/|Q(k-1)|)\int_{Q(k)}|f(x)-f_{Q(k)}|dx$$
$$\leq C\|f\|_{w,*}w(Q(k))/|Q(k)|\qquad(k=1,2,\cdots).$$

and

$$\int_{Q(k)}|f(x)-f_Q|dx\leq\int_{Q(k)}|f(x)-f_{Q(k)}|dx+|Q(k)||f_{Q(k)}-f_Q|$$
$$\leq\|f\|_{w,*}w(Q(k))+|Q(k)|\sum_{j=1}^{k}|f_{Q(j)}-f_{Q(j-1)}|$$
$$\leq C\|f\|_{w,*}[w(Q(k))+|Q(k)|\sum_{j=1}^{k}(w(Q(j))/|Q(j)|)]$$
$$\leq C\|f\|_{w,*}\sum_{j=1}^{k}2^{n(k-j)}w(Q(j)).$$

So

$$\int_{R^n}[|f(x)-f_Q|/(r^{np}+|x-x_0|^{np})]dx$$
$$\leq r^{-np}\int_Q|f(x)-f_Q|dx+\sum_{k=1}^{\infty}\int_{Q(k)\setminus Q(k-1)}[|f(x)-f_Q|/(r^{np}+|x-x_0|^{np})]dx$$
$$\leq r^{-np}\|f\|_{w,*}w(Q)+C\sum_{k=1}^{\infty}(2^kr)^{-np}\int_{Q(k)}|f(x)-f_Q|dx$$
$$\leq r^{-np}\|f\|_{w,*}w(Q)+C\sum_{k=1}^{\infty}\{(2^kr)^{-np}\sum_{j=1}^{k}[2^{(k-j)n}w(Q(j))]\}\|f\|_{w,*}.$$

Since $w\quad A_p(1<p<\infty)$,

$w(Q(j))\leq Cw(Q(j))\setminus Q(j-1))$ and $\int_{R^n}[w(x)/(r^{np}+|x-x_0|^{np})]dx\leq Cr^{n-np}w(Q)/|Q|$,

thus,

$$\sum\{\cdots\}=\sum_{k=1}^{\infty}(2^kr)^{-np}[2^{kn}\sum_{j=1}^{k}2^{-jn}w(Q(j))]$$
$$\leq C\sum_{k=1}^{\infty}\{\sum_{j=1}^{k}[2^{-(k-j)n(p-1)}\cdot(2^jr)^{-np}w(Q(j)\setminus Q(j-1))]\}$$
$$=C\sum_{j=1}^{\infty}\{\sum_{k=j}^{\infty}[2^{-n(k-j)(p-1)}(2^jr)^{np}w(Q(j)\setminus Q(j-1))]\}$$
$$\leq C\sum_{j=1}^{\infty}[(2^jr)^{-np}w(Q(j)\setminus Q(j-1))]$$
$$\leq C\sum_{j=1}^{\infty}\{\int_{Q(j)\setminus Q(j-1)}[w(x)/(r^{np}+|x-x_0|^{np})]dx\}$$

$$\leqslant C\int_{\mathbf{R}^n}[w(x)/(r^{np}+|x-x_0|^{np})]dx$$

$$\leqslant Cr^{n-np}w(Q)/|Q|.$$

Therefore,

$$\int_{\mathbf{R}^n}[|f(x)-f_Q|/(r^{np}+|x-x_0|^{np})|dx \leqslant r^{-np}\|f\|_{w,*}w(Q)+Cr^{n-np}\|f\|_{w,*}w(Q)/|Q|$$

$$\leqslant Cr^{n-np}\|f\|_{w,*}w(Q)/|Q|.$$

This completes the proof of the special case of the Lemma.

For the general case, arguing as in the proof of [10, Lemmal.1], we can finish the proof of the Lemma.

Proof (Theorem 1) Suppose $|\{g(f)\not\equiv\infty\}|>0$ and decompose f as in [10]

$$f(x)=f_1(x)+f_2(x)+f_3(x),$$

where $f_1(x)=f_Q$, $f_2(x)=[f(x)-f_Q]\chi_Q$ and $f_3(x)=[f(x)-f_Q]\chi_{C_Q}$

Obviously, $g(f_1)(x)\equiv 0$ and

$$(*)\quad g(f_2)\in L^1(Q) \text{ and } \int_Q|g(f_2)(x)|dx\leqslant C\|f\|_{w,*}w(Q).$$

In fact, since $w\in A$, and $f\in BMO_w$ by [1], we get

$$w(\{x\in Q:|f(x)-f_Q|w^{-1}>a\})\leqslant C\operatorname{Exp}(-ca/\|f\|_{w,*})w(Q).$$

Where C, c are constants independent of f. Thus,

$$\int_{\mathbf{R}^n}|f|^2w^{-1}dx=\int_Q|(f(x)-f_Q)w^{-1}|^2w(x)dx.$$

$$\leqslant C\int_0^\infty a\operatorname{Exp}(-ca/\|f\|_{w,*})w(Q)da\leqslant C\|f\|^2_{w,*}w(Q).$$

For $w\in A_1$, We have $w\in A_2$ and also $w^{-1}\in A_2$ from [5],

$$\|s(f_2)\|_{L^2(w^{-1}dx)}\leqslant C\|f_2\|_{L^2(w^{-1}dx)}.$$

From [6], $g(f_2)(x)\leqslant Cs(f_2)(x)$, hence,

$$\|g(f_2)\|_{L^2(w^{-1}dx)}\leqslant C\|s(f_2)\|_{L^2(w^{-1}dx)}\leqslant C\|f_2\|_{L^2(w^{-1}dx)}\leqslant C\|f\|_{w,*}[w(Q)]^{1/2}.$$

and

$$\int_Q|g(f_2)(x)|dx\leqslant C[w(Q)]^{1/2}\|g(f_2)\|_{L^2(w^{-1}dx)}\leqslant C\|f\|_{w,*}w(Q).$$

$(*)$ is proved.

Next, we want to show that for sufficiently small d depending only on n there is a constant C depending only on n such that for all $x\in dQ$,

(i) $g(f_3)(x')<\infty \Rightarrow g(f_3)(x)<\infty$,

(ii) $|g(f_3)(x)-g(f_3)(x')|\leqslant C\|f\|_{w,*}w(Q)/|Q|$,

where x' is a point in dQ as in [10].

Recall the proof of the corresponding results (i) and (ii) as in 17, Lemma 1, the estimates in [7] depend closely on the key property of BMO function:

$$(\blacktriangle)\quad \int_{\mathbf{R}^n}[|f(t)-f_{Qd}|/(d^{n+a}+|t-t_0|^{n+a})]dt\leqslant(A_n/d^a)\|f\|_*,$$

where Qd is a cube with centre t_0 and edge length d.

Repeat the argument of the proof of the corresponding estimates in [7] and use the preceding Lemma (Corollary) instead of ($\blacktriangle$), we can prove (i) and (ii). Arguing as in [10], $g(f)(x)<\infty$ a.e. is immediate.

Finally, we want to show that $\|g(f)\|_{w,*}\leq C\|f\|_{w,*}$.

Let Q' be any cube in R^n, set $Q=(1/d)Q'$ ($Q'=dQ$) and choose a point $x'\in dQ$ such that $g(f_2)(x')<\infty$. Then, by (*) and (ii), we get

$$\int_{Q'}|g(f)(x)-g(f_3)(x')|dx=\int_{Q'}|g(f_2+f_3)(x)-g(f_3)(x)+g(f_3)(x)-g(f_3)(x')|dx$$
$$\leq\int_{Q'}|g(f_2)(x)|dx+\int_{Q'}|g(f_3)(x')-g(f_3)(x')|dx$$
$$\leq\int_{Q}|g(f_2)(x)|dx+\int_{Q'}|g(f_3)(x)-g(f_3)(x')|dx \quad (\text{since } Q'\subset Q)$$
$$\leq C\|f\|_{w,*}w(Q)$$
$$\leq C\|f\|_{w,*}w(Q') \qquad (\text{since } w(Q)=w((1/d)Q)\leq Cw(Q')).$$

Since Q' is an arbitrary cube, we have $\|g(f)\|_{w,*}\leq C\|f\|_{w,*}$. The proof is completed for g-function.

For s-function $s(f)$ and g_λ^*-function $g_\lambda^*(f)$ ($\lambda>3$), arguing as in [8], noti noticing the proof of Theorems in [8] is also depending on (▲), we can readily prove the similar results as stated in Theorem 1. (Use the preceding Lemma instead)

Theorem 1 is completed.

§ 3 The proof of Theorem 2

Proof (Theorem 2) **By** [1, Theorem 5.], we claim that

(* *) If $f\in(\mathrm{BMO})_w$ and $w\in A_\infty$, then $f\in\mathrm{BMO}$,

and there exist constants C_1, C_2 independent of f such that

$$C_1\|f\|_{*,w}\leq\|f\|_*\leq C_2\|f\|_{*,w}.$$

In fact, if $f\in(\mathrm{BMO})_w$, according to the proof of [1, Theorem 5.1, we get

$$w(\{x\in Q:|f(x)-C_Q|>a\})\leq\exp(-ca/\|f\|_{*,w})w(Q),$$

where C, c are constants depending only on n.

Since $w\in A_\infty$, there are positive constants A, B, δ and η such that

$$A(|E|/|Q|)^\eta\leq w(E)/w(Q)\leq B(|E|/|Q|)^\delta,$$

which implies

$$|\{x\in Q:|f(x)-C_Q|>a\}|\leq C\mathrm{Exp}(-ca/(\eta\|f\|_{*,w}))|Q|,$$

and

$$(1/|Q|)\int_Q|f(x)-C_Q|\leq C\int_0^\infty\mathrm{Exp}(-ca/(\eta\|f\|_{*,w}))da\leq C_2\|f\|_{*,w}.$$

Thus $\|f\|_*\leq C_2\|f\|_{*,w}$.

Conversely, by $f\in\mathrm{BMO}$, we can easily get

$$f\in(\mathrm{BMO})_w \text{ and } \|f\|_{*,w}\leq C\|f\|_*.$$

The claim is true.

Use (* *) if $f\in(\mathrm{BMO})_w$, then $f\in\mathrm{BMO}$, $g(f)\not\equiv\infty$ a.e. from [7], $g(f)\in$ BMO, and

$$\|g(f)\| \leq C\|f\| \leq C\|f\|_{*,w}.$$

Using $(**)$ again, we have $g(f)\in(\mathrm{BMO})_w$ and

$$\|g(f)\|_{*,w}\leq C\|g(f)\|_{*}.$$

Therefore $\|g(f)\|_{*,w}\leq C\|f\|_{*,w}$.

The same argument for s-function $s(f)$, g_λ^*-function $g_\lambda^*(f)$ is also true. Theorem 2 is completed.

Remark 1

Clearly, Theorem 1 and 2 are the extensions of the corresponding results in [7] [8] [9] and [10].

Remark 2

For Marcinkiewicz integral $\mu(f)$ (see [9]), we also have the similar results as Theorem 1 and 2.

At the end of this paper, we shall give the following proposition analogous to the preceding Lemma in § 2.

Proposition If $w\in A_p(1<p<\infty)$, $f\in(\mathrm{BMO})_w$ and $\varepsilon>0$, then

$$\int_{R^n}[|f(x)-(f)_{Q,w}|/(r^{np+\varepsilon}+|x-x_0|^{np+\varepsilon})]w(x)\,dx\leq Cr^{-\varepsilon}\|f\|_{*,w}w(Q)/|Q|^p,$$

where Q is a cube centred at x_0 and having edge lenth r.

Especialy,

$$\int_{R^n}[|f(x)-(f)_{Q_0,w}|/(1+|x|^{n+\varepsilon})]w(x)\,dx\leq C\|f\|_{*,w}w(Q_0),$$

where Q_0 is the unit cube.

Proof (Proposition) Arguing similarly as in [3] for $k=1,2,\cdots$, we have

$$|(f)_{Q(x),w}-(f)_{Q(k-1),w}|\leq[1/w(Q(k-1))]\int_{Q(k-1)}|f(x)-(f)_{Q(k),w}|w(x)\,dx$$
$$\leq\|f\|_{*,w}w(Q(k))/w(Q(k-1)).$$

Since $w\in A_p$,

$$w(Q(k))=w(2Q(k-1))\leq C2^{np}w(Q(k-1))\leq Cw(Q(k-1))$$
$$w(Q(k))=w(2^k(Q))\leq C(2^k)^{np}w(Q),$$

thus $|(f)_{Q(k),w}-(f)_{Q(k-1),w}|\leq C\|f\|_{*,w}$, and

$$\int_{Q(k)}|f(x)-(f)_{Q,w}|w(x)\,dx$$
$$\leq\int_{Q(k)}|f(x)-(f)_{Q(k),w}|w(x)\,dx+w(Q(k))|(f)_{Q(k),w}-(f)_{Q,w}|$$
$$\leq CW(Q_{(k)})(1+k)\|f\|_{*,w}.$$

Therefore,

$$\int_{R^n}[|f(x)-(f)_{Q,w}|/(r^{np+\varepsilon}+|x-x_0|^{np+\varepsilon})]w(x)\,dx$$
$$\leq\int_{Q}[|f(x)-(f)_{Q,w}|/(r^{np+\varepsilon}+|x-x_0|^{np+\varepsilon})]w(x)\,dx+$$
$$+\sum_{k=1}^{\infty}\int_{Q(k)\setminus Q(k-1)}[|f(x)-(f)_{Q,w}|/(r^{np+\varepsilon}+|x-x_0|^{np+\varepsilon})]w(x)\,dx$$
$$\leq C(r^{-(np+\varepsilon)}\int_{Q}|f(x)-(f)_{Q,w}|w(x)\,dx+$$

$$+\sum_{k=1}^{\infty}(2^k r)^{-(np+\varepsilon)}\int_{Q(k)}|f(x)-(f)_{Q,w}|w(x)\mathrm{d}x]$$
$$\leqslant Cr^{-(np+\varepsilon)}\|f\|_{*,w}[w(Q)+\sum_{k=1}^{\infty}(2^k)^{-(np+\varepsilon)}(1+k)(2^k)^{np}w(Q)]$$
$$\leqslant Cr^{-(np+\varepsilon)}\|f\|_{*,w}[1+\sum_{k=1}^{\infty}2^{-k\varepsilon}(1+k)]w(Q)$$
$$\leqslant Cr^{-\varepsilon}\|f\|_{*,w}w(Q)/|Q|^p.$$

The proposition is completed .

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加权BMO函数空间上的Littlewood-Paley算子

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摘　　要

Littlewood—Paley 算子（g—函数，s—函数与 g_λ^*—函数，$\lambda>3$）作为BMO_w或$(\mathrm{BMO})_w$上的算子都是“有界的”,确切地说，我们证明了：若$f\in\mathrm{BMO}_w$或$(\mathrm{BMO})_w$且$|\{x:Tf(x)\neq\infty\}|>0$，则$Tf$也属于$\mathrm{BMO}_w$或$(\mathrm{BMO})_w$并且存在与$f$无关的常数$C$使$\|Tf\|_{\mathrm{BMO}_w}\leqslant C\|f\|_{\mathrm{BMO}_w}$($\|Tf\|_{(\mathrm{BMO})_w}\leqslant C\|f\|_{(\mathrm{BMO})_w}$)，其中$T$为Littlewood-Paley算子.