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A Kind of Extremal Problems*

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Abstract

A kind of extremal problems is discussed and characterizations of its solutions are given. The extremal problem is: Given $d_i(x_0, \cdots, x_n)$ for $i=0,\cdots,n$, to find $y_0<\cdots<y_n$ so that

$$f(y_0,\cdots,y_n)=\sup_{x_0<\cdots<x_n} f(x_0,\cdots,x_n),$$

where

$$f(x_0,\cdots,x_n):=\min_{0\le i\le n} d_i(x_0,\cdots,x_n)\Big/\sum_{j=0}^{n} d_j(x_0,\cdots,x_n).$$

I. Introduction

Let $d_i:=d_i(x_0,\cdots,x_n)$ with $x_0<\cdots<x_n$, $i=0,\cdots,n$, be given. Denote

$$d:=\sum_{i=0}^{n} d_i;$$

$$f_i:=d_i/d,\quad i=0,\cdots,n;$$

$$f(x_0,\cdots,x_n):=\min_{0\le i\le n} f_i(x_0,\cdots,x_n).$$

Our extremal problem is to find $y_0<\cdots<y_n$ so that

$$f(y_0,\cdots,y_n)=\sup_{x_0<\cdots<x_n} f(x_0,\cdots,x_n). \tag{1}$$

The main result of this paper is the following

Theorem Assume that for $i,j=0,\cdots,n$ and $s=1$ or -1, fixed,

$$d_i>0, \tag{2}$$

$$s\frac{\partial d_i}{\partial x_i}>0,\quad s\frac{\partial d_i}{\partial x_j}<0,\quad j\ne i, \tag{3}$$

$$\left|\frac{\partial d_i}{\partial x_i}\right|\ge\sum_{\substack{j=0\\ j\ne i}}^{n}\left|\frac{\partial d_i}{\partial x_j}\right|. \tag{4}$$

Then (1) is valid if and only if

$$f_i(y_0,\cdots,y_n)=f(y_0,\cdots,y_n)=1/(n+1),\quad i=0,\cdots,n. \tag{5}$$

The proof of the Theorem is more complicated and is given in Section 3. It needs several lemmas which we put in Section 2. Finally, in Section 4, we deduce some interesting corollaries.

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2. Lemmas

Lemma 1 If $A:=\det(a_{ij})_{i,j=1}^{n}$ has the form

$$a_{ij}=\begin{cases}1-a_i, & j=i\neq k\\ -a_i, & \text{otherwise}\end{cases}\qquad i,j=1,\cdots,n, \tag{6}$$

then $A=-a_k$.

Proof If $a_k=0$, then $A=0$ and the conclusion is true. Assume $a_k\neq 0$. For each $i\neq k$ add $-a_i/a_k$ times the k-th row to the i-th row to derive the new determinant $\det(b_{ij})_{i,j=1}^{n}$, where

$$b_{ij}=\begin{cases}1, & j=i\\ 0, & j\neq i\end{cases}\qquad i=1,\cdots,k-1,k+1,\cdots,n,\quad j=1,\cdots,n \tag{7}$$

$$b_{kj}=-a_k,\qquad j=1,\cdots,n.$$

It is easy to see that $A=\det(b_{ij})_{i,j=1}^{n}=-a_k$.

Lemma 2 If $A:=\det(a_{ij})_{i,j=1}^{n}$ has the form

$$a_{ij}=\begin{cases}1-a_i, & j=i\\ -a_i, & j\neq i\end{cases}\qquad i,j=1,\cdots,n, \tag{8}$$

then $A=1-\sum_{i=1}^{n}a_i$.

Proof By Lemma 1

$$A=\begin{vmatrix}-a_1 & -a_1 & \cdots & -a_1\\ -a_2 & 1-a_2 & \cdots & -a_2\\ \cdots & \cdots & \cdots & \cdots\\ -a_n & -a_n & \cdots & 1-a_n\end{vmatrix}+\begin{vmatrix}1 & 0 & \cdots & 0\\ -a_2 & 1-a_2 & \cdots & -a_2\\ \cdots & \cdots & \cdots & \cdots\\ -a_n & -a_n & \cdots & 1-a_n\end{vmatrix}=-a_i+\begin{vmatrix}1-a_2 & \cdots & -a_2\\ \cdots & \cdots & \cdots\\ -a_n & \cdots & 1-a_n\end{vmatrix}$$

$$=-a_1-\cdots-a_{n-1}+(1-a_n)=1-\sum_{i=1}^{n}a_i.$$

Lemma 3 If $A:=\det(a_{ij})_{i,j=1}^{n}$ satisfies that

$$|a_{ii}|>\sum_{j\neq i}|a_{ij}|,\quad i=1,\cdots,n, \tag{10}$$

then $\operatorname{sgn}A=\operatorname{sgn}\prod_{i=1}^{n}a_{ii}$.

Proof Add $-a_{1j}/a_{11}$ times the first column to the j-th column for $j=2,\cdots,n$ to derive the new determinant $B:=\det(b_{ij})_{i,j=1}^{n}$, here

$$b_{i1}=a_{i1},\ b_{ij}=a_{ij}-\frac{a_{i1}a_{1j}}{a_{11}},\quad i=1,\cdots,n,\quad j=2,\cdots,n.$$

But it from (10) follows that for $i\geq 2$

$$|b_{ii}|\geq|a_{ii}|-\left|\frac{a_{i1}a_{1i}}{a_{11}}\right|=|a_{ii}|-|a_{i1}|+|a_{i1}/a_{11}|(|a_{11}|-|a_{1i}|)$$

$$>\sum_{\substack{j=2\\ j\neq 1}}^{n}|a_{ij}|+|a_{i1}/a_{11}|\sum_{\substack{j=2\\ j\neq i}}^{n}|a_{1j}\geq\sum_{\substack{j=2\\ j\neq i}}^{n}\left|a_{ij}-\frac{a_{i1}a_{1j}}{a_{11}}\right|=\sum_{\substack{j=2\\ j\neq i}}^{n}|b_{ij}|,$$

which shows that the determinant $\det(b_{ij})_{i,j=2}^{n}$ of order $n-1$ satisfies (10), too.

On the other hand, since for $i \geq 2$, $|a_{11}| > |a_{1i}|$ and hence $|a_{ii}| > |a_{i1}| \geq |a_{1i} a_{i1}/a_{11}|$,

$$\operatorname{sgn} b_{ii} = \operatorname{sgn}(a_{ii} - \frac{a_{1i} a_{i1}}{a_{11}}) = \operatorname{sgn} a_{ii}.$$

By induction

$$\operatorname{sgn} A = \operatorname{sgn} B = \operatorname{sgn}(a_{11} \det(b_{ij})^n_{i,j=2}) = (\operatorname{sgn} a_{11}) \operatorname{sgn} \prod_{i=2}^{n} b_{ii} = \operatorname{sgn} \prod_{i=1}^{n} a_{ii}.$$

Lemma 4

$$s_{ij} := \operatorname{sgn} \frac{\partial(f_0, \cdots, f_{i-1}, f_{i+1}, \cdots, f_n)}{\partial(d_0, \cdots, d_{j-1}, d_{j+1}, \cdots, d_n)} = (-1)^{i-j}, \quad i, j = 0, \cdots, n. \tag{11}$$

Proof It is easy to see that

$$\frac{\partial f_k}{\partial d_r} = \begin{cases} (1 - f_k)/d, & r = k \\ -f_k/d, & r \neq k \end{cases} \quad k, r = 0, \cdots, n. \tag{12}$$

Let $j = i$. In this case the determinant $d^n \frac{\partial(f_0, \cdots, f_{i-1}, f_{i+1}, \cdots, f_n)}{\partial(d_0, \cdots, d_{i-1}, d_{i+1}, \cdots, d_n)}$ has the form (8) and by Lemma 2 is equal to $1 - \sum_{k \neq i} f_k = f_i$. Whence $s_{ii} = 1$.

Let $j \neq i$ and assume for definiteness that $i > j$. Note that

$$\frac{\partial(f_0, \cdots, f_{i-1}, f_{i+1}, \cdots, f_n)}{\partial(d_0, \cdots, d_{j-1}, d_{j+1}, \cdots, d_n)} = (-1)^{i-j-1} \frac{\partial(f_0, \cdots\cdots\cdots\cdots, f_{i-1}, f_{i+1}, \cdots, f_n)}{\partial(d_0, \cdots, d_{j-1}, d_i, d_{j+1}, \cdots, d_{i-1}, d_{i+1}, \cdots, d_n)}.$$

We may use Lemma 1 to compute the term on the right side of the above equation, which is equal to $(-1)^{i-j-1}(-f_i/d^n) = (-1)^{i-j} f_i/d^n$. Whence $s_{ij} = (-1)^{i-j}$.

Lemma 5

$$S_{ji} := \operatorname{sgn} \frac{\partial(d_0, \cdots, d_{j-1}, d_{j+1}, \cdots, d_n)}{\partial(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)} = (-1)^{i-j} s^n, \quad i, j = 0, \cdots, n. \tag{13}$$

Proof Let $i = j$. In this case the determinant

$$\frac{\partial(d_0, \cdots, d_{j-1}, d_{j+1}, \cdots, d_n)}{\partial(x_0, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)}$$

satisfies (10) because of (3) and (4), and is of the sign s^n by Lemma 3.

Let $i \neq j$ and assume for definiteness that $i > j$. In this case

$$\frac{\partial(d_0, \cdots, d_{j-1}, d_{j+1}, \cdots, d_n)}{\partial(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_a)} = (-1)^{i-j-1} \frac{\partial(d_0, \cdots, d_{j-1}, d_{j+1}, \cdots\cdots\cdots\cdots, d_n)}{\partial(x_0, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{i-1}, x_j, x_{i+1}, \cdots, x_n)}.$$

In the last determinant add each column except the i-th column $(\frac{\partial d_0}{\partial x_j}, \cdots, \frac{\partial d_{j-1}}{\partial x_j}, \frac{\partial d_{j+1}}{\partial x_j}, \cdots, \frac{\partial d_n}{\partial x_j})^T$ to it to derive a new determinant.

The elements of its i-th row are

$$\sum_{r \neq i} \frac{\partial d_i}{\partial x_r} \text{ and } \frac{\partial d_i}{\partial x_r}, \quad r \neq i, j$$

in which the first one lies on the diagonal. Thus by (3) and (4)

$$\left|\sum_{r\neq i}\frac{\partial d_i}{\partial x_r}\right|>\sum_{r\neq i,j}\left|\frac{\partial d_i}{\partial x_r}\right|.$$

Also, the elements of the row corresponding to the index $k\neq i,j$ are

$$\frac{\partial d_k}{\partial x_k},\ \sum_{r\neq i}\frac{\partial d_k}{\partial x_r}\ \text{and}\ \frac{\partial d_k}{\partial x_r},\ r\neq i,j,k$$

in which the first one lies on the diagonal. Thus by (3) and (4)

$$\left|\frac{\partial d_k}{\partial x_k}\right|=s\frac{\partial d_k}{\partial x_k}>s\frac{\partial d_k}{\partial x_k}+s\frac{\partial d_k}{\partial x_j}$$

$$=\sum_{r\neq i,j,k}\left(-s\frac{\partial d_k}{\partial x_r}\right)+\sum_{r\neq i}s\frac{\partial d_k}{\partial x_r}=\sum_{r\neq i,j,k}\left|\frac{\partial d_k}{\partial x_r}\right|+\left|\sum_{r\neq i}\frac{\partial d_k}{\partial x_r}\right|.$$

Therefore the new determinant satisfies (10) and by Lemma 3 is of the sign $s^{n-1}(-s)=-s^n$. Hence $S_{ji}=(-1)^{i-j}s^n$.

Lemma 6

$$\operatorname{sgn}\frac{\partial(f_0,\cdots,f_{i-1},f_{i+1},\cdots,f_n)}{\partial(x_0,\cdots,x_{i-1},x_{i+1},\cdots,x_n)}=s^n,\ i=0,\cdots,n. \tag{14}$$

Proof We see by Lemmas 4 and 5 that

$$\operatorname{sgn}\frac{\partial(f_0,\cdots,f_{i-1},f_{i+1},\cdots,f_n)}{\partial(x_0,\cdots,x_{i-1},x_{i+1},\cdots,x_n)}$$

$$=\operatorname{sgn}\sum_{j=0}^{n}\frac{\partial(f_0,\cdots,f_{i-1},f_{i+1},\cdots,f_n)}{\partial(d_0,\cdots,d_{j-1},d_{j+1},\cdots,d_n)}\cdot\frac{\partial(d_0,\cdots,d_{j-1},d_{j+1},\cdots,d_n)}{\partial(x_0,\cdots,x_{i-1},x_{i+1},\cdots,x_n)}=s^n.$$

3. Proof of the Theorem

Necessity Assume that (1) is true. Suppose on the contrary that there exists an index $i, 0\leq i\leq n$, such that

$$f_i(y_0,\cdots,y_n)>f(y_0,\cdots,y_n). \tag{15}$$

Since by Lemma 6 $\frac{\partial(f_0,\cdots,f_{i-1},f_{i+1},\cdots,f_n)}{\partial(x_0,\cdots,x_{i-1},x_{i+1},\cdots,x_n)}\neq 0$, for t, $0<t\leq(f_i(y_0,\cdots,y_n)-f(y_0,\cdots,y_n))/(n+1)$, small enough, there exist $x_0,\cdots,x_n$, $x_0<\cdots<x_n$ and $x_i=y_i$, such that

$$f_j(x_0,\cdots,x_n)=f_j(y_0,\cdots,y_n)+t,\qquad j=0,\cdots,i-1,i+1,\cdots,n.$$

Then

$$f_i(x_0,\cdots,x_n)=1-\sum_{j\neq i}f_j(x_0,\cdots,x_n)=1-\sum_{j\neq i}f_j(y_0,\cdots,y_n)-nt$$

$$=f_i(y_0,\cdots,y_n)+t-(n+1)t\geq f_i(y_0,\cdots,y_n)+t+f(y_0,\cdots,y_n)$$

$$-f_i(y_0,\cdots,y_n)=f(y_0,\cdots,y_n)+t.$$

This means that $f(x_0,\cdots,x_n)>f(y_0,\cdots,y_n)$, a contradiction.

Sufficiency This is trivial, because one always has $f\leq 1/(n+1)$ and (5) implies (1).

4. Corollaries

By a similar argument we may obtain the following.

Corollary 1 Under the assumptions of the Theorem

$$\max_{0\le i\le n} f_i(y_0,\cdots,y_n)=\inf_{x_0<\cdots<x_n}\ \max_{0\le i\le n} f_i(x_0,\cdots,x_n)$$

if and only if (5) is valid.

Now we give a modified result which is as follows.

Corollary 2 Let x_k be fixed. Then under the assumptions of the Theorem for $y_0<\cdots<y_n$ with $y_k=x_k$

$$f(y_0,\cdots,y_n)=\sup_{\substack{x_0<\cdots<x_n\\ x_k\ \text{fixed}}} f(x_0,\cdots,x_n)$$

if and only if (5) is valid.

Proof The sufficiency is trivial. In order to establish the necessity suppose on the contrary that there exists an index i, $0\le i\le n$, such that (15) occurs. If $i=k$ then the same argument as above is still suitable. If $i\ne k$, then for t as above there exist $x_0,\cdots,x_n$, $x_0<\cdots<x_n$ and x_k fixed, such that

$$f_j(x_0,\cdots,x_n)=\begin{cases} f_j(y_0,\cdots,y_n)+t, & j\ne i\\ f_i(y_0,\cdots,y_n)-nt, & j=i\end{cases}\qquad j=0,\cdots,k-1,\ k+1,\cdots,n.$$

Thus $f_k(x_0,\cdots,x_n)=1-\sum_{j\ne k} f_j(x_0,\cdots,x_n)=f_k(y_0,\cdots,y_n)+t$ and one may again use the same argument as above.

The existence of the functions d_i's can be seen from the following.

Proposition Let $a_{ij}\ge b_{ij}>0$ for $i,j=0,\cdots,n$ and let $g_i(x)$, $i=0,\cdots,n$, satisfy that

$$g_i>0,\ sg_i'>0,\quad i=0,\cdots,n \text{ and } s=1 \text{ or } -1 \text{ fixed}.$$

Then

$$d_i=\sum_{\substack{j=0\\ j\ne i}}^{n} g_j(a_{ij}x_i-b_{ij}x_j),\quad i=0,\cdots,n$$

satisfy (2)—(4).

Proof Clearly

$$d_i>0,\quad i=0,\cdots,n.$$

$$s\frac{\partial d_i}{\partial x_i}=s\sum_{j\ne i} a_{ij}g_j'(a_{ij}x_i-b_{ij}x_j)>0,\quad i=0,\cdots,n,$$

$$s\frac{\partial d_i}{\partial x_j}=-sb_{ij}g_j'(a_{ij}x_i-b_{ij}x_j)<0,\quad j\ne i,\quad i,j=0,\cdots,n,$$

and

$$\left|\frac{\partial d_i}{\partial x_i}\right|=\sum_{j\ne i} a_{ij}\left|g_j'(a_{ij}x_i-b_{ij}x_j)\right|\ge\sum_{j\ne i} b_{ij}\left|g_j'(a_{ij}x_i-b_{ij}x_j)\right|\ge\sum_{j\ne i}\left|\frac{\partial d_i}{\partial x_j}\right|.$$

一 类 极 值 问 题

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摘 要

本文讨论了一类极值并刻划了它的解的特征. 该极值问题为：给定 $d_i(x_0, \cdots, x_n)$，$i=0, \cdots, n$，要确定 $y_0 < \cdots < y_n$ 使得

$$f(y_0, \cdots, y_n = \sup_{x_0 < \cdots < x_n} f(x_0, \cdots, x_n)$$

其中

$$f(x_0, \cdots, x_n) := \min_{0 \le i \le n} d_i(x_0, \cdots, x_n) \Big/ \sum_{j=0}^{n} d_j(x_0, \cdots, x_n).$$