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Note on "Some New Results of P-Injective Rings" and "Regular Rings Are Very Regular"*

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In this note, we point out and correct some mistakes in [1] and [2] that are related to (von Neumann) regular rings.

Throughout R is an associative ring with identity and $E(R)=\{e|e=e^2\in R\}$. Following [1], we call $E(R)$ weakly closed in case for any $e, f\in E(R)$ there exists $g\in E(R)$ with $Ref=Rg$. It should be noted that the weakly closed-ness is left-right symmetric, since for any $r\in R$: Rr is a summand of $_RR$ if and only if rR is a summand of R_R In [1], the authors incorrectly claimed that $E(R)$ is weakly closed for any ring R. Consequently, they induced the incorrect [1, Proposition 5] and [1, Theorem 6] (= [2 Theorem]). The next example gives an explanation.

Example 1 Let F be a field and $R=\begin{bmatrix} F & F \\ 0 & F\end{bmatrix}$. Choose idempotents $e=\begin{bmatrix} 1 & 1 \\ 0 & 0\end{bmatrix}$ and $f=\begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}$. Then $Ref=R\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}$ is not a summand of R, and so $E(R)$ is not weakly closed. Clearly $Ref\subsetneqq Re$, so the equality in [1, p.9, line −4] is incorrect. In general, we only have $Ref\oplus Re(1-f)\supseteq Re$. To show that [1, Proposition 5] is not correct, we take an idempotent $h=\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}$. Then $Re+Rh=\left\{\begin{bmatrix} a & b \\ 0 & 0\end{bmatrix} \middle| a, b\in F\right\}$ is not a summand of R.

Recall that R is a regular ring in case $r\in rRr$ for any $r\in R$. It is well-known that R is regular if and only if every principal left (right) ideal of R is a summand of R. Recently, Tjukavkin in [3, Example 3] has constructed a non-regular ring R such that each finitely generated right ideal of R can be generated by finitely many idempotents. This shows that [1, Theorem 6] and [2, Theorem] are not correct.

Using a result in [4], we observe the following

Lemma 2 Let R be a ring and $e, f\in E(R)$. The following two statemente

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are equivalent :

(1) There exists $h \in E(R)$ such that $Rf(1-e)=Rh$.

(2) There exists $g \in E(R)$ such that $Rg=Re+Rf$.

Proof One notes that $Re+Rf=Re \oplus Rf(1-e)$.

(1)⇒(2). We have $he=0$, so it follows from [4, p.102, Exercise 6(1)] that $e+h-eh \in E(R)$ and $R(e+h-eh)=Re+Rh=Re+Rf(1-e)=Re+Rf$.

(2)⇒(1). Since $Rg=Re+Rf=Re \oplus Rf(1-e)$ and Rg is a summand of R, so $Rf(1-e)$ is also a summand of R.

The above lemma enables us to correct [1, Proposition 5] as follows.

Proposition 3 If $E(R)$ is weakly closed then for any $e, f \in E(R)$ there exists $g \in E(R)$ such that $Re+Rf=Rg$.

It is easy to see that $E(R)$ is weakly closed if R is a regular ring. Using the above result and the induction, we now correct [1, Theorem 6] and [2, Theorem].

Theorem 4 A ring R is regular if and only if each finitely generated left ideal of R can be generated by finitely many idempotents and $E(R)$ is weakly closed.

Remark 5 (1) A ring R is called normal in case each idempotent of R lies in the center of R. For a normal ring R, $E(R)$ is closed under multiplication and so $E(R)$ is weakly closed;

(2) As noted earlier, $E(R)$ is weakly closed if R is regular;

(3) According to Example 1, the above result can not be extended to semihereditary rings;

(4) If $R=\pi R_i$ is an arbitrary product of rings, then $E(R)$ is weakly closed if and only if each $E(R_i)$ is weakly closed.

Our concluding proposition gives a criterion for $E(R)$ to be weakly closed.

Proposition 6 If R is a ring and $e, f \; E(R)$, then Ref is a summand of R if and only if Ref is projective and any homomorphism from Ref to R can be extended to one from R to R.

Proof (⇒). This direction is obvious.

(⇐). Since Ref is projective, there exists $g \in \mathrm{Hom}_R(Ref, R)$ with $ef=g(ef)ef$. By hypothesis, there exists $h \in \mathrm{Hom}_R(R, R)$ with $h|_{Ref}=g$. Let $r=h(1)$, then

$$ef=g(ef)ef=h(ef)ef=efh(1)ef=efref.$$

Hence $ref \in E(R)$, and $Ref=Rref$ is a summand of R.

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References

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关于“P—内射环的几个新结果”和“正则环是非常正则的”两文的注记

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摘　要

本文指出并且修正了文献“P—内射环的几个新结果”和“正则环是非常正则的”的几个错误.