Journal of Mathematical Research & Exposition,
Vol.12, No.2, May 1992

Uniform Approximation by Multidimensional Szász-Mirakjan Operators*

Zhou Dingxuan *Xuan Peicai* *Zhang Zhenqiu*
(Dept. of Math., Zhejiang University, Hangzhou, China)

1. Introduction

Let $S_n(f;x)$ denote the Szász-Mirakjan operators:

$$S_n(f;x)=\sum_{k=0}^{\infty} f(k/n)P_{nk}(x), \tag{1}$$

where $P_{nk}(x)=e^{-nx}\frac{(nk)^k}{k!},\ k\geq 0,\ n,k\in N_0$.

Recently, many authors studied the Szász-Mirakjan operators and their versions [1–5]. In this paper, we shall deal with the multidimensional case. We give our results only for two variables, the higher dimensional problem can be treated similarly. We will use the decomposition technique introduced by D.X. Zhou [8,9]for multidimensional operators on simplex as well as on cubes.

Let $S_{n,m}(f;x,y)$ be the two variable Szasz-Mirakjan operators [3]:

$$S_{n,m}(f;x,y)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} f(k/n,l/m)P_{nk}(x)P_{ml}(y), \tag{2}$$

where $n,m\in N$.

In the case where n,m are independent, V.Totik [3] proved that:

For $0<\alpha\leq 1, f\in C_{2B}(T)$, the following statements are equivalent:

$$\begin{cases} \text{(i)} & \| S_{n,m}(f)-f \|_{\infty}=O(n^{-\alpha}+m^{-\alpha}), \\ \text{(ii)} & w(\delta_1,\delta_2)=O(\delta_1^{2\alpha}+\delta_2^{2\alpha}), \end{cases} \tag{3}$$

where $C_{2B}(T)$ denotes the set of continuous and bounded functions defined on the first quadrant

$$T=\{(x,y)||x\geq 0,y\geq 0\},$$
$$w(\delta_1,\delta_2)=\sup_{0<h_1\leq\delta_1,0<h_2\leq\delta_2} \| \Delta^2_{h_1\sqrt{x},h_2\sqrt{y}}(f;x,y) \|_{L_\infty(T)};$$
$$\Delta^2_{h_1,h_2}(f;x,y)=f(x,y)+f(x,y+2h_2)+f(x+2h_1,y)$$
$$+f(x,2h_1y+2h_2)-4f(x+h_1,y+h_2).$$

*Received May 22, 1990. Research supported by Zhejiang Provincial Natural Science Foundation.

In this paper, we shall consider the case when n, m are dependent on each other and satisfy the following condition

$$k_1 \le \frac{n}{m} \le k_2, \tag{4}$$

where k_1, k_2 are some positive constants.

Our main result is the following:

Theorem 1.1 *Let $f \in C_{2B}(T), 0 < \alpha < 1$. Then the following statements are equivalent:*

$$\begin{cases} (1) & K(f(,t) = O(t^\alpha); \\ (2) & \| S_{n,m}(f) - f \|_\infty = O(n^{-\alpha}); \\ (3) & \begin{cases} \text{(a)} & \| f(x + t\varphi(x), y) - 2f(x,y) + f(x - t\varphi(x), y) \|_\infty = O(t^{2\alpha}), \\ \text{(b)} & \| f(x, y + h\varphi(y)), -2f(x,y) + f(x, y - h\varphi(y)) \|_\infty = O(t^{2\alpha}). \end{cases} \end{cases} \tag{5}$$

where $k(f,t)$ is the $k-$functional

$$K(f,t) = \inf_{g \in D}\{\| f - g \|_\infty + tN(g)\},$$

and D is the weighted Sobolev spaces:

$$D = \{f \mid f \in C_{2B}(T), f_{11}, f_{22} \text{ exists and } N(f) < \infty\},$$

where

$$N(f) = \| sf_{11}(s,t) \|_\infty + \| tf_{22}(s,t) \|_\infty$$

and

$$f_{11} \triangleq \frac{\partial^2 f(s,t)}{\partial s^2}, \ f_{22} \triangleq \frac{\partial^2 f(s,t)}{\partial t^2}, \ \varphi(x) = \sqrt{x}, \varphi(y) = \sqrt{y}.$$

2. Lemmas

To prove Theorem 1.1, we need some lemmas.

Lemma 2.1 *For $f \in C_{2B}(T)$,*

$$\| S_{n,m}(f; x, y) \|_\infty \le \| f \|_\infty.$$

Proof It is trivial from the defintion (2) and

$$\| S_n(f; x) \|_\infty \le \| f \|_\infty.$$

Lemma 2.2 *For $f \in D$,*

$$\| S_{n,m}(f) - f \|_\infty \le (\frac{M}{n} + \frac{M}{m})N(f).$$

Where M denotes a constant independent of n, m and f.

Proof For $f(x,y) \in C_{2B}(T)$, let

$$f_x(y) = f(x,y), y \in [0,\infty), x \quad \text{fixed};$$
$$f^y(x) = f(x,y), x \in [0,\infty), \ y \quad \text{fixed}.$$

Then we have

$$\begin{aligned} S_{n,m}(f) - f &= S_{n,m}(f(s,t) - f_s(y) + f^y(s) - f(x,y); x,y) \\ &= S_{n,m}(f(s,t) - f_s(y); x,y) + S_{n,m}(f^y(s) - f(x,y); x,y) \\ &\triangleq I + J. \end{aligned}$$

From the results for the univariable operaators in [3, 5] we have

$$\begin{aligned} |I| &= |S_n(S_m(f_s;y) - f_s(y)); x)| \le \sup_s |S_m(f_s;y) - f_s(y)| \\ &\le \frac{M}{m}\sup_s \|\varphi^2 f_s''\| = \frac{M}{m}\sup_s \|tf_{22}(s,t)\|_{[0,\infty)} \\ &= \frac{M}{m}\|tf_{22}(s,t)\|_{[0,\infty)\times[0,\infty)}; \end{aligned} \tag{6}$$

$$\begin{aligned} |J| &= |S_{n,m}(f^y(s) - f^y(x); x,y)| = |S_n(f^y(s) - f^y(x); x)| \\ &= |S_n(f^y;x) - f^y(x)| \le \frac{M}{n}\|\varphi^2 (f^y)''\|_\infty \\ &\le \frac{M}{n}\|sf_{11}(s,y)\|_{[0,\infty)\times[0,\infty)}. \end{aligned}$$

Thus we have

$$\|S_{n,m}(f) - f\|_\infty = \sup_{x,y}|S_{n,m}(f;x,y) - f(x,y)|$$
$$\le \left(\frac{M}{n} + \frac{M}{m}\right)(\|sf_{11}(s,t)\|_\infty + \|tf_{22}(s,t)\|_\infty) = \left(\frac{M}{n} + \frac{M}{m}\right)N(f). \quad \Box$$

Lemma 2.3 *For $f \in C_{2B}(T)$, then*

$$N(S_{n,m}(f)) \le Mn\|f\|_\infty.$$

Where M is a constant independent of n, m and f.

Proof By [3,5], we have

$$\begin{aligned} \left\|x\frac{\partial^2}{\partial}x^2 S_{n,m}(f;x,y)\right\|_\infty &= \left\|x\sum_{k,l} f\left(\frac{k}{n},\frac{l}{m}\right)P_{ml}(y)P_{nk}''(x)\right\|_\infty \\ &\le \left\|x\sum_{k=0}^{\infty}\|f\|_\infty P_{nk}''(x)\right\|_\infty \le Mn\|f\|_\infty. \end{aligned}$$

Similarly, we can estimate

$$\| y\frac{\partial^2}{\partial y^2}S_{n,m}(f;x,y)\|_\infty \le Mn\| f \|_\infty.$$

The result follows from this and (4).

Lemma 2.4 *For $f \in D$,*

$$N(S_{n,m}(f)) \le MN(f),$$

where M is independent of n, m and f.

Proof We have

$$\begin{aligned}
| x(S_{n,m}(f)_{11}(x,y) | &= | \sum_{l=0}^{\infty} P_{ml}(y)x\sum_{k=0}^{\infty} f^{l/m}(\frac{k}{n})P''_{nk}(x) | \\
&= | \sum_{l=0}^{\infty} P_{ml}(y)\{xS_n(f^{1/m}(\cdot);x)''\} | \le M\sum_{l=0}^{\infty} P_{ml}(y)\| \varphi^2(f^{1/m}(\cdot))'' \|_\infty \\
&\le M\sum_{l=0}^{\infty} P_{ml}(y)\sup_s | sf_{11}(s,\frac{l}{m}) | \le M\sup_{0\le t<\infty}\{\sup_s | sf_{11}(s,t) |\} \\
&= M\| sf_{11} \|_\infty;
\end{aligned}$$

Similarly, we can obtain $\| y(S_{n,m}(f)_{22}(x,y) \| \le M\| tf_{22} \|_\infty$, this completesthe proof. □

3. Proof of Theorem 1.1

By a result of [6] and from Lemmas 2.1–2.4, we have the equivalence of (i) and (ii). Next, we prove (i)$\Longrightarrow$ (iii). a) For $g(x,y) \in D$, we have

$$\begin{aligned}
& | f(x-h\varphi(x),y) - 2f(x,y) + f(x+h\varphi(x),y) | \\
\le\ & 4\| f-g \|_\infty + | g(x-h\varphi(x),y) - 2g(x,y) + g(x+h\varphi(x),y) | \\
=\ & 4\| f-g \|_\infty + \int_{-\frac{h}{2}\varphi(x)}^{\frac{h}{2}\varphi(x)}\int_{-\frac{h}{2}\varphi(x)}^{\frac{h}{2}\varphi(x)} \frac{\partial^2}{\partial x^2}g(x+s+t)\,ds\,dt \\
\le\ & 4\| f-g \|_\infty + \| \varphi^2(x)g_{11}(x,y) \|_\infty \int\int_{-\frac{h}{2}\varphi(x)}^{\frac{h}{2}\varphi(x)} \frac{ds\,dt}{x+s+t} \\
\le\ & 4\| f-g \|_\infty + 8h^2\| \varphi^2(x)g_{11}(x,y) \|_\infty.
\end{aligned}$$

Taking the infimum of the right-hand side over all $g \in D$,we have

$$\| f(x-h\varphi(x),y) - 2f(x,y) + f(x+h\varphi(x),y) \|_\infty \le 8K(f,h^2) \le ch^{2\alpha}.$$

Similarly, we can obtain iii b) of (3). Therefore (i)$\Longrightarrow$(iii) follows.

For (iii)$\Longrightarrow$(ii), using the decomposition technique, we write

$$S_{n,m}(f) - f = S_{n,m}(f(s,) - f_s(y);x,y) + S_{n,m}(f^y(s) - f(x,y);x,y) \triangleq I + J.$$

Then, by the results of [3,7], we have

$$\begin{aligned}\| I \|_\infty &= \| S_n(S_m(f_s, y) - f_s(y), x) \|_\infty \le \sup_{0\le s<\infty} \sup_{0\le y<\infty} | S_m(f_s, y) - f_s(y) | \\ &\le M \sup_S w_\varphi^2(f_s; \frac{1}{\sqrt{n}}).\end{aligned}$$

Where, M is independent of n, s and f.

$$w^2(f_s; \frac{1}{\sqrt{n}}) = \sup_{0<t\le\frac{1}{\sqrt{n}}} | f_s(z + t\varphi(z)) - 2f_s(z) + f_s(z - t\varphi(z) | .$$

Therefore,

$$\begin{aligned}\| I \|_\infty &\le M \sup_{0\le s<\infty} \sup_{0<t\le\frac{1}{\sqrt{n}}} | f(s, z - t\varphi(z)) - 2f(s,z) + f(s, z + t\varphi(z)) | \\ &\le M \sup_{0\le s<\infty} \sup_{0<t\le\frac{1}{\sqrt{n}}} t^{2\alpha} \le Cn^{-\alpha}.\end{aligned}$$

By the same method, we can obtain

$$\| J \|_\infty \le Cn^{-\alpha}.$$

Therefore we have

$$\| S_{n,m}(f) - f \|_\infty = O(n^{-\alpha}).$$

This completes the proof of Theorem 1.1. □

Acknowledgment We would like to thank our advisor—Prof.Guo Zhurui for his careful guidance and constant encouragement.

References

[1] M. Becker, *Global approximation theorems for Szasz- Mirakjan and Baskakov operators in polynomial weight spaces*, Indian Univ. Math. J., bf 27(1978), 127–142.

[2] M. Becker, D. Kucharski and R.J. Nessel, *Global approximation theorems for the Szasz-Mirakjan operators in exponential weight spaces*, in *Linear Spaces and Approximation*, Birkhauser, Verlag, Basel, 1977.

[3] V. Totik, *Uniform approximation by Szaszx-Mirakjan type operators*, Acta. Math. Hung., bf 41:3–4(1983), 291–307.

[4] V. Totik, *Approximation by Szasz-Mirakjan-Kantorovich operators in $L^p(p > 1)$*, Analysis Mathematica, bf 9(1983), 147–167.

[5] Zhou Dingxuan, *Approximation by Szasz-Mirakjan type operator in L^p*, Adavances in Math., bf 19:1(1990), 118–120.

[6] A. Grundmann, *Inverse theorems for Kantorovich polynomials,* Fourier Analysis and Application Theory Proc. Conf., Budapest, 1976, 395–401.

[7] Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer Series in Computational Math., Vol.9, Springer-Verlag, 1987,

[8] Zhou Dingxuan, *Inverse theorems for multidimensional Bernstein-Durmeyer operators in L_p*, (to appear).

[9] Zhou Dingxuan, *Weighted approximation by multidimensional Bernstein operators*, (to appear).

多元 Szász—Mirkjan 算子的一致逼近

周定轩　　宣培才　　张震球

（浙江大学数学系）

摘　　要

本文研究了多元 Szása—Mirakjan 算子在 $C_{2B}(T)$ 中的逼近性质，利用 K—泛函，建立了等价的逼近定理. 主要结果如下

定理　设 $f\in C_{2B(T)}$，$0<\alpha<1$，则以下命题等价

(i)　$K(f,t)=0(t^{\alpha})$；

(ii)　$\| S_{n,m}(f)-f\|_{\infty}=0(n^{-\alpha})$；

(iii)　a)　$\| f(x+t\varphi(x),y)-2f(x,y)+f(x-t\varphi(x),y)\|_{\infty}=0(t^{2\alpha})$；

b)　$\| f(x,y+h\varphi(y))-2f(x,y)+f(x,y-h\varphi(y))\|_{\infty}=0(h^{2\alpha})$.