Journal of Mathematical Research & Exposition,
Vol.12, No.2, May 1992

The Approximation of Continuous Function by Kantorovitch Operator and Its Transformation Operator *

Ye Nanfa
(Zhangzhou Teachers' College, Fujian)

Kantorovitch polynomials operator is the following extension of Bernstein polynomials operator

$$K_n[f(t):x]=(n-1)\sum_{k=0}^{n}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f(t)dt\right)C_n^kx^k(1-x)^{n-k}.$$

Transformation operator or $K_u[f(t):x]$ is defined by

$$\overline{k}_n[f(t):x]=\sum_{k=0}^{n}C_n^kx^k(1-x)^{n-k}\frac{1}{2}[f(\frac{k}{n-1})^{n-k}+f(\frac{k+1}{n+1})]$$

According to [2],for any $f(x)\in C_{[0,1]}$ then

$$|\ K_n[f(t):x]-f(x)\ |\le\frac{5}{4}\omega(f,\frac{1}{\sqrt{n+1}}),\tag{1}$$

$$|\ \overline{k}_n[f(t):x]-f(x)\ |\le\frac{5}{4}\omega(f,\frac{1}{\sqrt{n+1}}),\tag{2}$$

where $\omega(f,\cdot)$ is the modulus of continuity of function $f(x)$.

Lemma 1 i) *The following identity holds*

$$\begin{aligned}L_n(x)&=\sum_{k=0}^{n}[(5k^4+10k^3+10k^2+5k+1)\\&-(20k^3+30k^2+20k+5)(n+1)x+(30k^2+30k+10)(n+1)^2x^2\\&-(20k+10)(n+1)^3x^3+5(n+1)^4x^4]C_n^kx^k(1-x)^{n-k}\\&=5nx(1-x)[20x^2-20x+5+3nx(1-x)]+(1-x)^5+x^5;\end{aligned}$$

ii) $L_n(x)\le\frac{15}{16}(n+1)^2$ (when $n>6,\ x\in[0,1]$).

Lemma 2 i) *The following identity holds*

$$\begin{aligned}\sigma_n(x)&=\sum_{k=0}^{n}[(n+1)x-k]^4C_n^kx^k(1-x)^{n-k}\\&=nx(1-x)[20x^2-10x+1+3nx(1-x)]+x^4;\end{aligned}$$

*Received Jun. 23. 1990.

ii) $\sigma_n(x) \le \frac{25}{128}(n+1)^2$ (when $n > 17,\ x \in [0,1]$).

Lemma 3 i) *The follwoing identity holds*

$$\begin{aligned}\overline{\sigma}_n(x) &= \sum_{k=0}^{n}[(n+1)x-(k+1)]^4 C_n^k x^k(1-x)^{n-k} \\ &= nx(1-x)[20x^2-30x+11+3nx(1-x)]+(1-x)^4;\end{aligned}$$

ii) $\overline{\sigma}_n(x) \le \frac{7}{32}(n+1)^2$ (when $n > 15,\ x \in [0,1]$).

By virtue of Lemmas, we show that (1) and (2) can be imoroved as the following results:

Theorom 1 *For any $f(x) \in C_{[0,1]}$ the following approximation holds when $n > 6$*

$$| K_n[f(t);x] - f(x) | \le \frac{19}{16}\omega(f, \frac{1}{\sqrt{n+1}}).$$

Theorem 2 *For any $f(x) \in C_{[0,1]}$ then*

$$| \overline{K}_n[f(t);x] - f(x) | \le \frac{309}{256}\omega(f, \frac{1}{\sqrt{n+1}}) \quad (\text{when } n > 17).$$

References

[1] Wang R.H.,*Approximation of Unbouded Function.* Sclence Rublishing House, China, 1983, in Chinese.

[2] Lee Ween-ching, Xiamen University Journal (Natural Science Edition), 1(1962), 71-73.