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A Note on Queueing System $M/M/1$*

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Abstract. A new formula is given for the queue length distribution functions of queueing system $M/M/1$. Based on this formula and renewal theory, a new and simple proof of the ergodicity of queueing system $M/M/1$ is given.

1. Introduction

The transient behavior of queueing system $M/M/1$ has been studied by many authors. The distribution of the queue length at instant t has been given out. The classical solution is based on the generating function and the formula is expressed by Bessel functions (see Hsu [2] or Saaty [4]). In Tian [1], an interesting formula for the distribution of the queue length at instant t for queueing system $M/M/1$ is given in a simple way. But the result given in [1] is only for queueing system $M/M/1$ initially empty. In this paper, we generalize Tain's result to queueing system $M/M/1$ with arbitrary initial conditions (see section 2). Then, we give an interesting explanation for the formula of queue length distribution functions and use the result to give a simple proof for the ergodic theorem for queueing system $M/M/1$ (see section 3).

2. Distribution of Queue Length

Queueing system $M/M/1$ has an input of Poisson process with parameter λ; the service time is of exponential distribution with parameter μ. There are one waiting line and one server in the system. The input process is independent of the services.

Let $q(t)$ be the queue length in the system at instant t; $P_n(t) = P\{q(t) = n\}$, $n \geq 0$. Then $\{P_n(t)\}$ satisfies the following differential equations (see Hsu [2], Chapter 2, or Saaty [4])

$$\begin{cases} P_0'(t) = -\lambda P_0(t) + \mu P_1(t), \\ P_n'(t) = \lambda P_{n-1}(t) - (\lambda+\mu)P_n(t) + \mu P_{n+1}(t), \ n \geq 1. \end{cases} \tag{1}$$

Given the initial condtion as $P_n(0) = 0, n \neq i$, $P_i(0) = 1$, the solution of (1) has been given as follows

$$P_n(t) = \left(\frac{\lambda}{\mu}\right)^{\frac{n-i}{2}} e^{-(\lambda+\mu)t}\left[I_{n-i}(2t\sqrt{\lambda\mu}) + \left(\frac{\lambda}{\mu}\right)^{-\frac{1}{2}} I_{n+i+1}(2t\sqrt{\lambda\mu})\right]$$

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$$+\left(\frac{\lambda}{\mu}\right)^{\frac{n-i}{2}} e^{-(\lambda+\mu)t}\left[(1-\frac{\lambda}{\mu})\sum_{k=1}^{\infty}(\frac{\lambda}{\mu})^{-\frac{k+1}{2}} I_{k+n+i+1}(2t\sqrt{\lambda\mu})\right], \quad n \geq 0. \tag{2}$$

This expression is very complex and ambiguous in probabilistic significance. In sequel, we will give a new formula for $\{P_n(t)\}$. Denote

$$\begin{cases} G_{-1}(t) \equiv 0, G_n(t) = e^{-\mu t}\sum_{j=0}^{n}\frac{(\mu t)^j}{j!}, n \geq 0, t \geq 0; \\ \Phi_n(t) = e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}G_{n+k}(t), n \geq -1, t \geq 0, \end{cases} \tag{3}$$

where $\rho = \frac{\lambda}{\mu}$ is the service intensity.

Theorem 1 *For queueing system $M/M/1$, if $P_n(0) = 0, n \neq i, P_i(0) = 1$, then*

$$\begin{aligned} P_n(t) &= (1-\rho)\rho^n - \rho^{\max\{0,n-i\}}\left[\Phi_{|n-i|-1}(t) - \Phi_{|n-i|}(t)\right] \\ &- \rho^n\left[\Phi_{n+i}(t) - \rho\Phi_{n+i+1}(t)\right], n \geq 0. \end{aligned} \tag{4}$$

Proof The theorem can be proved by showing that $\{P(n,t)\}$ given by (4) satisfy (1) (see Tian [1] for the case $P_0(0) = 1$). We omit the proof here.

Obviously there are two advantages of (4) over (2). a): It is expressed by simple combinations of simple functions. b): The limitation of $P_n(t)$ is separate from the part of $P_n(t)$ related to t, which gives a better understanding of the solution of equation (1).

3. Probabilistic Significance

Let $N(t)$ and $S(t)$ be the renewal numbers in $[0,t]$ for the Poisson processes with parameters λ and μ respectively. The two processes are independent. Then the following properties are held.

$$\begin{aligned} &1): G_{-1}(t) = P\{S(t) \leq -1\} \equiv 0, \\ &\quad G_n = P\{S(t) \leq n\} = 1 - \int_0^t \frac{\mu^{n+1}t^n}{n!}e^{-\mu t}\,dt, n \geq 0; \\ &2): \Phi_n(t) = P\{S(t) - N(t) \leq n\}, n \geq -1, \\ &3): \Phi_n(t) - \Phi_{n-1}(t) = P\{S(t) - N(t) = n\}, n \geq 0. \end{aligned} \tag{5}$$

By (5), (4) can be rewritten in the following interesting form:

$$\begin{aligned} P_n(t) &= \rho^{\max\{0,n-i\}}P\{S(t) - N(t) = \mid n-i \mid\} \\ &+ \rho^n\left[P\{S(t) - N(t) > n+i\} - \rho P\{S(t) - N(t) > n+i+1\}\right], n \geq 0. \end{aligned} \tag{6}$$

For renewal process $N(t)$(or $S(t)$), $\lim \frac{N(t)}{t} = \lambda$, w.p.l. ($\lim \frac{S(t)}{t} = \mu$, w.p.l)(see [5], pp.58).

1) If $\rho < 1$, i.e., $\lambda < \mu$, then

$$\lim_{t\to\infty}(S(t) - N(t)) = \lim_{t\to\infty} t\left[\frac{S(t)}{t} - \frac{N(t)}{t}\right] = +\infty, \quad \text{w.p.l;} \tag{7}$$

2) If $\rho > 1$, i.e., $\lambda > \mu$, then

$$\lim_{t\to\infty}(S(t) - N(t)) = -\infty, \quad \text{w.p.l}; \tag{8}$$

3) If $\rho = 1$, i.e., $\lambda = \mu$, let $X_n = S(n) - N(n) - [S(n-1) - N(n-1)]$, $n \geq 1$. Since $S(t)$ and $N(t)$ are poisson processes, $\{X_n\}$ are i.i.d. random variables with $EX_n = 0$. Denote $S_0 = 0, S_n = \sum_{i=1}^n X_n, n \geq 1$. By Chung and Fuchs [3], it is easy to prove

$$\liminf_{t\to\infty}\{S(t) - N(t)\} \leq \liminf_{n\to\infty}\{S_n\} = -\infty, \quad \text{w.p.l}, \tag{9}$$

and

$$\limsup_{t\to\infty}\{S(t) - N(t)\} \geq \limsup_{n\to\infty}\{S_n\} = +\infty, \quad \text{w.p.l} \tag{10}$$

Theorem 2 *For queueing system $M/M/1$, it is positive recurrent if and only if $\rho < 1$. And*

$$\lim_{t\to+\infty} P_n(t) = \begin{cases} (1-\rho)\rho^n, & \text{if } \rho < 1; \\ 0, & \text{if } \rho \geq 1. \end{cases}$$

Proof If $\rho < 1$, by (6) and (7),

$$\lim_{t\to\infty} P\{S(t) - N(t) =| n - i |\} = 0, \ \lim_{t\to\infty} P\{S(t) - N(t) > n + i\} = 1.$$

So

$$\lim_{t\to\infty} P_n(t) = (1-\rho)\rho^n, n \geq 0.$$

If $\rho \geq 1$, by (6), (8), (9) and (10),

$$\lim_{t\to\infty} P_n(t) = 0, n \geq 0.$$

This completes the proof. □

To end this paper, we guess that this method can be used to queueing systems $GI/M/1$ and $M/G/1$, to give iterative algorigthms for the distribution of the queue length at the arriving instants or departure instants of customers.

References

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