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范畴Fun, Nat和Adi*

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§1 引　言

自然变换讨论了一类特殊的函子，一般函子之间的关系如何描述呢？例如：$\forall$ 函子$F : C$ D $\exists!$ 函子$F^* : C/R_1 \to D/R_2$, 使得有交换图(如右), 其中R_1, R_2 为C, D 范畴中的合同关系，满足条件：$\forall f, g \in \mathrm{Arr}\,C(fR_1g \supset FfR_2Fg)$. 而$Q_{R_1}, Q_{R_2}$ 为泛函子. 显然，函子F 与F^* 之间的关系不是自然变换所能处理的.

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "Q_{R_1}"'] & D \arrow[d, "Q_{R_2}"] \\ C/R_1 \arrow[r, "F^*"'] & D/R_2 \end{tikzcd}$$

对此，本文引进了函子间态的概念，建立了任意函子组成的范畴Fun, 并证明了它是带积范畴. 由此可知，从任何两个函子出发，我们都可以构造出它们的积.

在数学的不同的领域，存在着许多性质各异的函子，利用积我们可构造出许多新的函子，这些函子在代数拓扑，同调代数理论中将有着广泛的应用. 例如Hom 函子与Ext 函子的积，Ext 函子与Tor 函子的积. 更进一步地，Hom, Ext, Tor 三者之积，所有的这些结果无疑会加深人们对原有的结构的认识. 同样的，从一般范畴上的泛性图我们就可构造出乘积范畴上的泛性图，例如：在Set 中任两推出图，即可得出Set^2 中的推出图. 从范畴C 中一个拉回图和致等图，就可构造出C^2 中一个新的泛性图.

按照这种思想本文进一步讨论了一切自然变换所成范畴Nat 及其积的性质. 这就为从近代微分几何众多的自然变换中构造新的变换创造了条件. 文[1] 中曾讨论了附加间态的概念，我们把它发展成为附加范畴Adi, 并证明了其带积性，它与Adi 从不同的侧面描述了附加的整体性质.

最后给出了Adi, Nat, Fun 和Cat 结构间的关系.

§2 函子范畴Fun

定义1 一个范畴称为是函子范畴Fun, 如果:

(1) 对象：函子$F, G, H, N, I, \ldots$;

(2) 矢量：若存在函子M, N, 使得有交换图(右图).

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "M"'] & B \arrow[d, "N"] \\ C \arrow[r, "G"'] & D \end{tikzcd}$$

则定义$\langle M, N\rangle$ 为函子F 到G 的一个态，记为:
$\langle M, N\rangle : F \to G. \forall F : A \to B$; 定义：$\langle 1_A, 1_B\rangle : F \to F$ 为单位态，其中$1_A : A \to A, 1_B : B \to B$ 为恒等函子. 而$\forall \langle M, N\rangle : F \to G, \langle H, L\rangle : G \to E$; 定义复合运算为：$\langle H, L\rangle\langle M, N\rangle = \langle HM, LN\rangle : F \to E$.

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定义的合理性是显然的. 而且这种函子间的态相当普遍，如当GF复合有意义时，$\langle F,G\rangle: F\to G$.

定理1 函子范畴Fun 是带有限积的范畴.

$$\begin{tikzcd} B & B\times C \arrow[l, "P"'] \arrow[r, "Q"] \arrow[d, dashed, "H"] & C \arrow[d, "V"] \\ B' & B'\times C' \arrow[l, "P'"'] \arrow[r, "Q'"] & C' \end{tikzcd}$$

证明 $\forall$ 函子U,V, 考虑到Cat 是带积范畴，从而右交换图唯一确定函子H, 其中P,Q,P',Q' 均为投影函子.

（上图中 $B \xrightarrow{U} B'$.）

根据范畴积的泛性知: $\forall$ 函子$G:E\to F,\langle M,N\rangle:G\to U,\langle S,T\rangle:G\to V,\exists!\langle R,K\rangle:G\to H$ 使得有交换图:

$$\begin{tikzcd} & E \arrow[dl, "M"'] \arrow[d, dashed, "R"] \arrow[dr, "S"] & \\ B \arrow[d, "U"'] & B\times C \arrow[l, "P"'] \arrow[r, "Q"] \arrow[d, dashed, "H"] & C \arrow[d, "V"] \\ B' & B'\times C' \arrow[l] \arrow[r] & C' \\ & F \arrow[ul, "N"] \arrow[u, dashed, "K"'] \arrow[ur, "T"'] & \end{tikzcd} \qquad (*)$$

注意: 由$NG=UM,TG=VS,M=PR,S=QR,N=P'K,T=Q'K$ 知$P'KG=NG=UM=UPR=P'HR$ 同理可证, $Q'KG=Q'HR$, 因此$KG=HR$. 即: $\langle R,K\rangle:G\to H$. 亦即$\exists\langle R,K\rangle:G\to H$, 使得有交换图如右，而且由(*) 的泛性易证得$\langle R,K\rangle$ 是唯一的，所以H为U 与V 的积，以后总记$H=U\times V$.

$$\begin{tikzcd} U & & H \arrow[ll, "{\langle P,P'\rangle}"'] \arrow[lld, "{\langle Q,Q'\rangle}"] \\ V & & G \arrow[ll, "{\langle S,T\rangle}"] \arrow[llu, "{\langle M,N\rangle}"'] \arrow[u, dashed, "{\langle R,K\rangle}"'] \end{tikzcd}$$

下面列举函子积的一些基本性质:

1. $\forall$ 函子$U:B\to B',V:C\to C',b\in \mathrm{Ob}B,c\in\mathrm{Ob}C$,有$:(U\times V)(b,c)=(Ub,Vc)$, 对态亦然.

2. $(U\times V)\times W\cong U\times(V\times W)$.

3. $(E\times G)\cdot(F\times H)=(EF)\times(GH)$, 特别地，$I_A\times I_B=I_{A\times B}$, 即恒等函子积为恒等函子.

4. 若F_1,F_2 都是守信(满) 函子，则$F_1\times F_2$ 亦然.

5. 若$\eta:E\dot\to F,\varepsilon:G\dot\to H$, 则有自然变换$\eta\times\varepsilon=((\eta\times\varepsilon)_{(e,g)})_{(e,g)\in\mathrm{Ob}(E\times G)}:E\times G\dot\to F\times H$.
   其中: $(\eta\times\varepsilon)_{(e,g)}=(\eta_e,\varepsilon_g)$.

6. 若$\eta,\eta',\varepsilon,\varepsilon'$ 为自然变换且下面的复合有意义，则$(\eta\times\varepsilon)\cdot(\eta'\times\varepsilon')=(\eta\eta')\times(\varepsilon\varepsilon')$.

7. 若 $\eta:E\simeq F,\varepsilon:G\simeq H$, 则$\eta\times\varepsilon:E\times G\simeq F\times H$.

8. 若E,G 为范畴等价(同构函子), 则$E\times G$ 亦为范畴等价(同构函子).

9. 若η_x 为x 到F 的泛矢，ε_y 为y 到G 的泛矢，则(η_x,ε_y) 为(x,y) 到$F\times G$ 的泛矢.

10 . 若$\langle E_1,F_1,\eta_1,\varepsilon_1\rangle:X_1\rightharpoonup A_1,\langle E_2,F_2,\eta_2,\varepsilon_2\rangle:X_2\rightharpoonup A_2$, 则有

$$\langle E_1\times E_2,F_1\times F_2,\eta_1\times\eta_2,\varepsilon_1\times\varepsilon_2\rangle:X_1\times X_2\rightharpoonup A_1\times A_2.$$

等价地，若$\langle F_1,G_1,\varphi_1\rangle:X_1\rightharpoonup A_1,\langle F_2,G_2,\varphi_2\rangle:X_2\rightharpoonup A_2$，则$\langle F_1\times F_2,G_1\times G_2,\varphi_1\times\varphi_2\rangle:X_1\times X_2\rightharpoonup A_1\times A_2$.

11. 若$\langle C_1,\mu_1:F_1\to\Delta_2C_1\rangle$为$F_1$到$\Delta_1$之上极限，$\langle C_2,\mu_2,F_2\to\Delta_2C_2\rangle$为$F_2$到$\Delta_2$之上极限，则$\langle\langle C_1,C_2\rangle,\mu_1\times\mu_2:F_1\times F_2\to\Delta\langle C_1,C_2\rangle\rangle$为$F_1\times F_2$到$\Delta$之极限. 其中$\Delta_1:C_1\to C_1^I,\Delta_2:C_2\to C_2^J,\Delta:C_1\times C_2\to(C_1\times C_2)^{I\times J}$都是对角函子. 更有：$(\lim_{\to}F_1,\lim_{\to}F_2)=\lim_{\to}(F_1\times F_2)$. 这在构造泛性图是有用的. 如Set中有拉回图：

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\mu"] \\ C \arrow[r, "v"'] & D \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "m"] \arrow[d, "n"'] & F \arrow[d, "s"] \\ G \arrow[r, "t"'] & H \end{tikzcd}$$

可得Set^2中的拉回图(右下图).

12. 若函子F,G可表示，则$F\times G$亦然.

$$\begin{tikzcd} (A,E) \arrow[r, "(f,m)"] \arrow[d, "(g,n)"'] & (B,F) \arrow[d, "(u,s)"] \\ (C,G) \arrow[r, "(v,t)"'] & (D,H) \end{tikzcd}$$

13. $\forall$函子$F:A\to B,G:C\to D$，则$F\times G:A\times C\to B\times D$. 设投影函子$P:B\times D\to B,Q:B\times D\to D$，则得双函子$S=P(F\times G):A\times C\to B,T=Q(F\times G):A\times C\to D$，且$\forall c\in\mathrm{Ob}C,S(-,c)=F;\forall a\in\mathrm{Ob}A,T(a,-)=G$. 特别地，设函子$F:A\to C,G:B\to C$，则$\exists$双函子$S,T:A\times B\to C$，使得$S(-,b)=F,T(a,-)=G$.

14. 给定函子$F:X\times P\to A,G:A\to X$，则存在双函子$S:P\times A\to X$，使得$S(p,-)=G,\forall p\in\mathrm{Ob}P$.

§3 自然变换范畴Nat

定义2 设$m:F\dot\to G,n:M\dot\to N$，若存在函子间的态$\langle F_1,F_2\rangle:F\to M,\langle G_1,G_2\rangle:G\to N$，使得有交换图：

$$\begin{tikzcd} \cdot \arrow[r, "m"] \arrow[d, "F_1"'] & \cdot \arrow[d, "G_2"] \\ \cdot \arrow[r, "n"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "m"] \arrow[d, "G_1"'] & \cdot \arrow[d, "F_2"] \\ \cdot \arrow[r, "n"'] & \cdot \end{tikzcd}$$

即：$G_2m=n_{F_1},F_2m=n_{G_1}$成立，则称有如右交换图.

$$\begin{tikzcd} F \arrow[r, "m"] \arrow[d, "\langle F_1,F_2\rangle"'] & G \arrow[d, "\langle G_1,G_2\rangle"] \\ M \arrow[r, "n"'] & N \end{tikzcd}$$

定义3 一个范畴称为是自然变换范畴Nat，如果(1) 对象：自然变换$m,n,u,v,\cdots$；(2) 矢量：若存在函子间态$\langle F_1,F_2\rangle,\langle G_1,G_2\rangle$使得有交换图(如右)，则定义$\langle\langle F_1,F_2\rangle,\langle G_1,G_2\rangle\rangle$为$m$到$n$的一个态，记为$\langle\langle F_1,F_2\rangle,\langle G_1,G_2\rangle\rangle:m\dot\to n.\forall m:F\dot\to G$，其中函子$F,G:X\to A$，定义$\langle\langle 1_X,1_A\rangle,\langle 1_X,1_A\rangle\rangle$为单位态，其中$1_X$，$1_A$为恒等函子. $\forall\langle\langle F_1,F_2\rangle,\langle G_1,G_2\rangle\rangle:m\to n,\langle\langle K_1,K_2\rangle,\langle L_1,L_2\rangle\rangle:n\to\mu$，定义复合运算为：

$$\begin{tikzcd} F \arrow[r, "m"] \arrow[d, "\langle F_1,F_2\rangle"'] & G \arrow[d, "\langle G_1,G_2\rangle"] \\ M \arrow[r, "n"'] & N \end{tikzcd}$$

$$\langle\langle K_1,K_2\rangle,\langle L_1,L_2\rangle\rangle\cdot\langle\langle F_1,F_2\rangle,\langle G_1,G_2\rangle\rangle=\langle\langle K_1,K_2\rangle\cdot\langle F_1,F_2\rangle,\langle L_1,L_2\rangle\cdot\langle G_1,G_2\rangle\rangle:m\dot\to\mu.$$

定义的合理性是容易验证的.

定理2 自然变换范畴Nat是带有限积范畴.

证明 $\forall m: F \to G, n: M \to N$, 有$m \times n: F \times M \to G \times N$, 且有交换图:

$$\begin{tikzcd} F \times M \arrow[r, "m \times n"] \arrow[d, "\langle P_1, P_2\rangle"'] & G \times N \arrow[d, "\langle P_1, P_2\rangle"] \\ F \arrow[r, "m"'] & G \end{tikzcd} \qquad \begin{tikzcd} F \times M \arrow[r, "m \times n"] \arrow[d, "\langle Q_1, Q_2\rangle"'] & G \times N \arrow[d, "\langle Q_1, Q_2\rangle"] \\ M \arrow[r, "n"'] & N \end{tikzcd}$$

其中$P_1: A \times C \to A, Q_1: A \times C \to C, P_2: B \times D \to B, Q_2: B \times D \to D$ 均为投影函子, $F: A \to B, M: C \to D$. 下证:

$$\begin{tikzcd} m & \\ & m \times n \arrow[ul, "\langle\langle P_1, P_2\rangle, \langle P_1, P_2\rangle\rangle"'] \arrow[dl, "\langle\langle Q_1, Q_2\rangle, \langle Q_1, Q_2\rangle\rangle"] \\ n & \end{tikzcd} \tag{**}$$

具有泛性. $\forall u: H \to L, \langle\langle R_1, R_2\rangle, \langle R_1', R_2'\rangle\rangle: u \to m, \langle\langle T_1, T_2\rangle, \langle T_1', T_2'\rangle\rangle: u \to n$, 由下面两泛性图

$$\begin{tikzcd} F & F \times M \arrow[l, "\langle P_1, P_2\rangle"'] \arrow[dl, "\langle Q_1, Q_2\rangle"] \\ M & H \arrow[u, dashed, "\langle S_1, S_2\rangle"'] \arrow[ul, "\langle R_1, R_2\rangle"] \arrow[l, "\langle T_1, T_2\rangle"] \end{tikzcd} \qquad \begin{tikzcd} G & G \times N \arrow[l, "\langle P_1, P_2\rangle"'] \arrow[dl, "\langle Q_1, Q_2\rangle"] \\ N & L \arrow[u, dashed, "\langle S_1', S_2'\rangle"'] \arrow[ul, "\langle R_1', R_2'\rangle"] \arrow[l, "\langle T_1', T_2'\rangle"] \end{tikzcd}$$

$$\begin{tikzcd} F \times M \arrow[r, "m \times n"] & G \times N \\ H \arrow[u, "\langle S_1, S_2\rangle"] \arrow[r, "u"'] & L \arrow[u, "\langle S_1', S_2'\rangle"'] \end{tikzcd}$$

知$\exists! \langle S_1, S_2\rangle, \langle S_1', S_2'\rangle$ 使得上述图形可交换. 下证有交换图(右图). 由条件知:

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "\langle R_1, R_2\rangle"'] & \cdot \arrow[d, "\langle R_1', R_2'\rangle"] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "\langle T_1, T_2\rangle"'] & \cdot \arrow[d, "\langle T_1', T_2'\rangle"] \\ \cdot \arrow[r, "n"'] & \cdot \end{tikzcd}$$

即: $m_{R_1} = R_2' u, R_2 u = m_{R_1'}, n_{T_1} = T_2' u, T_2 u = n_{T_1'}$. 又由$R_1 = P_1 S_1, R_2 = P_2 S_2, T_1 = Q_1 S_1, T_2 = Q_2 S_2, R_1' = P_1 S_1', R_2' = P_2 S_2', T_1' = Q_1 S_1', T_2' = Q_2 S_2'$, 得:

$$m_{P_1 S_1} = P_2 S_2' u, P_2 S_2 u = m_{P_1 S_1'}, n_{Q_1 S_1} = Q_2 S_2' u, Q_2 S_2 u = n Q_1 S_1'.$$

因此

$$S_2' u = (P_2 S_2' u, Q_2 S_2' u) = (m_{P_1 S_1}, n_{Q_1 S_1}) = (m \times n)_{(P_1 S_1, Q_1 S_1)} = (m \times n)_{S_1}.$$

同理$S_2 u = (m \times n)_{S_1'}$, 从而有$\langle\langle S_1, S_2\rangle, \langle S_1', S_2'\rangle\rangle: u \to m \times n$. 亦即$\exists \langle\langle S_1, S_2\rangle, \langle S_1', S_2'\rangle\rangle$, 使得有交换图:

$$
\begin{tikzcd}
m & \\
 & m\times n \arrow[ul, "{\langle\langle P_1,P_2\rangle,\langle P_1,P_2\rangle\rangle}"'] \arrow[dl, "{\langle\langle Q_1,Q_2\rangle,\langle Q_1,Q_2\rangle\rangle}"] \\
n & \\
 & u \arrow[u, dashed, "{\langle\langle S_1,S_2\rangle,\langle S_1',S_2'\rangle\rangle}"'] \arrow[uuul, bend left, "{\langle\langle R_1,R_2\rangle,\langle R_1',R_2'\rangle\rangle}"] \arrow[ul, "{\langle\langle T_1,T_2\rangle\langle T_1',T_2'\rangle\rangle}"]
\end{tikzcd}
$$

且易证得唯一性，所以(**) 具有泛性. 以后记m 与n 积为$m\times n$.

下面列举自然变换积的一些基本性质:

1. 若$m:F\to G, n:M\to N$, 则$m\times n:F\times M\to G\times N$. 对于自然同构亦然.

2. $\forall F,G\in \mathrm{Ob\ Fun}, m,n\in \mathrm{Ob\ Nat}$, 有$Fm\times Gn=(F\times G)(m\times n)$.

3. $\forall m_1,m_2,m_3,m_4\in \mathrm{Ob\ Nat}$, 只要复合有意义，则

$$(m_1\times m_2)\circ(m_3\times m_4)=(m_1\circ m_3)\times(m_2\circ m_4).$$

4. 设f_*,g_* 分别为f,g 诱导的自然变换，则$f_*\times g_*$ 为由$\langle f,g\rangle$ 诱导的自然变换.

5. $f_*\times g_*$ 分别为f,g 诱导的自然变换，则$f_*\times g_*$ 匹的当且仅当f,g 皆为满的; $f_*\times g_*$ 满的当且仅当f,g 皆为可裂单的; $f_*\times g_*$ 同构当且仅当f,g 皆为同构态.

§4 附加范畴Adi

定义4 一个范畴称为是附加范畴Adi, 如果:

(1) 对象: 附加$\langle F_1,G_1,\varphi_1\rangle,\langle F_2,G_2,\varphi_2\rangle,\cdots$;

(2) 矢量: 若函子间态$\langle H,L\rangle:F_1\to F_2$ 满足: (a) $\langle L,H\rangle:G_1\to G_2$; (b) 有交换图(右图). $\forall x_1\in \mathrm{Ob}X_1, a_1\in\mathrm{Ob}A_1$ 成立. 其中$F_1:X_1\to A_1, F_2:X_2\to A_2$, 则称$\langle H,L\rangle$ 为附加$\langle F_1,G_1,\varphi_1\rangle$, 和$\langle F_2,G_2,\varphi_2\rangle$ 间的态，记为$\langle H,L\rangle:\langle F_1,G_1,\varphi_1\rangle$ 和$\langle F_2,G_2,\varphi_2\rangle$. 且定义复合运算为$\langle H,L\rangle\langle M,N\rangle=\langle HM,LN\rangle$.$\forall$ 附加$\langle F,G,\varphi\rangle:X\to A$, 取$\langle 1_X,1_A\rangle$ 为单位态，其中$1_X,1_A$ 为恒等函子.

$$
\begin{tikzcd}
A_1(F_1x_1,a_1) \arrow[rr, "\varphi_1"] \arrow[d, "L"'] & & X_1(x_1,G_1a_1) \arrow[d, "H"] \\
A_2(LF_1x_1,La_1) \arrow[d, equal] & & X_2(Hx_1,HG_1a_1) \arrow[d, equal] \\
A_2(F_2Hx_1,La_1) \arrow[rr, "\varphi_2"] & & X_2(Hx_1,G_2La_1)
\end{tikzcd}
$$

定义的合理性是明显的.

定理3 附加范畴Adi 是带有限积的范畴.

证明 今设$\langle F_1,G_1,\varphi_1\rangle:X_1\to A_1$, $\langle F_2.G_2,\varphi_2\rangle:X_2\to A_2$, 则有$\langle F_1\times F_2,G_1\times G_2,\varphi_1\times\varphi_2\rangle:X_1\times X_2\to A_1\times A_2$. 显然有交换图(如右图和下图).

$$
\begin{tikzcd}
X_1\times X_2 \arrow[d, "P_1"'] & A_1\times A_2 \arrow[l, "F_1\times F_2"'] \arrow[r, "G_1\times G_2"] \arrow[d, dashed, "P_2"] & X_1\times X_2 \arrow[d, "P_1"] \\
X_1 & A_1 \arrow[l, "F_1"] \arrow[r, "G_1"'] & X_1
\end{tikzcd}
$$

$$\begin{tikzcd}
(A_1 \times A_2)(F_1 \times F_2-,-) \arrow[r, "\varphi_1 \times \varphi_2"] \arrow[d, "P_2"'] & (X_1 \times X_2)(-,G_1 \times G_{2-}) \arrow[d, "P_1"] \\
A_1(P_2(F_1 \times F_2)_-,P_{2-}) \arrow[r, "\varphi_1"'] & X_1(P_{1-},P_1(G_1 \times G_2)_-)
\end{tikzcd}$$

因此$\langle P_1,P_2\rangle:\langle F_1\times F_2,G_1\times G_2,\varphi_1\times\varphi_2\rangle\to\langle F_1,G_1,\varphi_1\rangle$. 类似地

$$\langle Q_1,Q_2\rangle:\langle F_1\times F_2,G_1\times G_2,\varphi_1\times\varphi_2\rangle\to\langle F_2,G_2,\varphi_2\rangle.$$

下证:

$$\begin{tikzcd}
\langle F_1,G_1,\varphi_1\rangle & \\
& \langle F_1\times F_2,G_1\times G_2,\varphi_1\times\varphi_2\rangle \arrow[ul, "\langle P_1,P_2\rangle"'] \arrow[dl, "\langle Q_1,Q_2\rangle"] \\
\langle F_2,G_2,\varphi_2\rangle &
\end{tikzcd}$$

具有泛性. $\forall\langle F,G,\varphi\rangle\in \mathrm{Ob\ Adi},\langle M,N\rangle:\langle F,G,\varphi\rangle\to\langle F_1,G_1,\varphi_1\rangle,\langle K,L\rangle:\langle F,G,\varphi\rangle\to\langle F_2,G_2,\varphi_2\rangle$. 由泛性图(下右图).

$$\begin{tikzcd}
F_1 & & \\
& & F_1\times F_2 \arrow[ull, "\langle P_1,P_2\rangle"'] \arrow[dll, "\langle Q_1,Q_2\rangle"] \\
F_2 & & F \arrow[uull, "\langle M,N\rangle"] \arrow[ll, "\langle K,L\rangle"] \arrow[u, dashed, "\langle S,T\rangle"']
\end{tikzcd}
\qquad
\begin{tikzcd}
G_1 & & \\
& & G_1G_2 \arrow[ull, "\langle P_2,P_1\rangle"'] \arrow[dll, "\langle Q_2,Q_1\rangle"] \\
G_1\times G_2 & & G \arrow[uull, "\langle N,M\rangle"] \arrow[ll, "\langle L,K\rangle"] \arrow[u, dashed, "\langle U,W\rangle"']
\end{tikzcd}$$

知$\exists!\langle S,T\rangle:F\to F_1\times F_2$ 使得上右图可换. 同样地由泛性图(上左图) 知$\exists!\langle U,W\rangle:G\to G_1\times G_2$, 使得上左图可换. 即有:

$$M=P_1S=P_1W,K=Q_1S=Q_1W,N=P_2T=P_2U,L=Q_2T=Q_2U.$$

从而易得: $S=W,T=U$. 即: $\langle T,S\rangle:G\to G_1\times G_2$. 下证有交换图:

$$\begin{tikzcd}
A(F-,-) \arrow[r, "\varphi"] \arrow[d, "T"'] & X(-,G-) \arrow[d, "S"] \\
(A_1\times A_2)(TF-,T-) \arrow[r, "\varphi_1\times\varphi_2"'] & (X_1\times X_2)(S-,SG-)
\end{tikzcd}$$

$\forall x\in \mathrm{Ob}X,a\in\mathrm{Ob}A,f\in A(Fx,a)$, 由$P_1S=M,Q_1S=K$ 知$P_1S\varphi f=M\varphi f,Q_1S\varphi f=K\varphi f$, 所以$S\varphi f=\langle P_1S\varphi f,Q_1S\varphi f\rangle=\langle M\varphi f,K\varphi f\rangle$, 同理$Tf=\langle Nf,Lf\rangle$. 再注意有交换图:

$$
\begin{tikzcd}
A(F-,-) \arrow[r, "\varphi"] \arrow[d, "N"'] & X(-,G-) \arrow[d, "M"] \\
A_1(NF-,N-) \arrow[r, "\varphi_1"'] & X_1(M-,MG-)
\end{tikzcd}
\qquad
\begin{tikzcd}
A(F-,-) \arrow[r, "\varphi"] \arrow[d, "L"'] & X(-,G-) \arrow[d, "K"] \\
A_2(LF-,L-) \arrow[r, "\varphi_2"'] & X_2(K-,KG-)
\end{tikzcd}
$$

因此

$$S\varphi f=\langle M\varphi f, K\varphi f\rangle=\langle\varphi_1 Nf,\varphi_2 Lf\rangle=(\varphi_1\times\varphi_2)\langle Nf,Lf\rangle=(\varphi_1\times\varphi_2)Tf.$$

所以$S\varphi=(\varphi_1\times\varphi_2)T$, 即上述图形可换. 故$\exists\langle S,T\rangle:\langle F,G,\varphi\rangle\to\langle F_1\times F_2,G_1\times G_2,\varphi_1\times\varphi_2\rangle$使得下图可换:

$$
\begin{tikzcd}
\langle F_1,G_1,\varphi_1\rangle & \\
 & \langle F_1\times F_2,G_1\times G_2,\varphi_1\times\varphi_2\rangle \arrow[ul, "{\langle P_1,P_2\rangle}"'] \arrow[dl, "{\langle Q_1,Q_2\rangle}"] \\
\langle F_2,G_2,\varphi_2\rangle & \\
 & \langle F,G,\varphi\rangle \arrow[uuul, "{\langle M,N\rangle}"] \arrow[ul, "{\langle K,L\rangle}"] \arrow[uu, dashed, "{\langle S,T\rangle}"']
\end{tikzcd}
$$

唯一性易得. 即$\langle F_1\times F_2,G_1\times G_2,\varphi_1\times\varphi_2\rangle$为$\langle F_1,G_1,\varphi_1\rangle$与$\langle F_2,G_2,\varphi_2\rangle$之积, 记成: $\langle F_1,G_1,\varphi_1\rangle\times\langle F_2,G_2,\varphi_2\rangle$. 下面列举附加积的一些基本性质:

1. 在复合有意义时, 成立:

$$
\begin{aligned}
&(\langle F_1,G_1,\varphi_1\rangle\times\langle F_2,G_2,\varphi_2\rangle)\cdot(\langle\overline{F_1},\overline{G_1},\overline{\varphi_1}\rangle\times\langle\overline{G_1},\overline{G_2},\overline{\varphi_2}\rangle)\\
&=(\langle F_1,G_1,\varphi_1\rangle\cdot\langle\overline{F_1},\overline{G_1},\overline{\varphi_1}\rangle)\times(\langle F_2,G_2,\varphi_2\rangle\cdot\langle\overline{F_2},\overline{G_2},\overline{\varphi_2}\rangle).
\end{aligned}
$$

2. 设$\langle H,L\rangle:\langle F,G,\varphi\rangle\to\langle F',G',\varphi'\rangle,\langle\overline{H},\overline{L}\rangle:\langle\overline{F},\overline{G},\overline{\varphi}\rangle\to\langle\overline{F}',\overline{G}',\overline{\varphi}'\rangle$, 则有

$$\langle H\times\overline{H},L\times\overline{L}\rangle:\langle F,G,\varphi\rangle\times\langle\overline{F},\overline{G},\overline{\varphi}\rangle\to\langle F',G',\varphi'\rangle\times\langle\overline{F}',\overline{G}',\overline{\varphi}'\rangle.$$

3. 若$\langle F_1,G_1,\varphi_1\rangle,\langle F_2,G_2,\varphi_2\rangle$都是伴随等价, 则$\langle F_1,G_1,\varphi_1\rangle\times\langle F_2,G_2,\varphi_2\rangle$亦然.

最后我们指出, 上述三种范畴间存在着满函子:

$$
\begin{tikzcd}
\mathrm{Adi} \arrow[r, dash, "U_1"] & \mathrm{Fun} \\
\langle F_1,G_1,\varphi_1\rangle \arrow[r] \arrow[d, "{\langle H,L\rangle}"'] & F_1 \arrow[d, "{\langle H,L\rangle}"] \\
\langle F_2,G_2,\varphi_2\rangle \arrow[r] & F_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Nat} \arrow[r, dash, "U_2"] & \mathrm{Fun} \\
F_1 \xrightarrow{\varphi_1} G_1 \arrow[r] \arrow[d, "{\langle\langle H,L\rangle,\langle M,N\rangle\rangle}"'] & F_1 \arrow[d, "{\langle H,L\rangle}"] \\
F_2 \xrightarrow[\varphi_2]{} G_2 \arrow[r] & F_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Fun} \arrow[r, dash, "U_3"] & \mathrm{Cat} \\
A_1 \xrightarrow{F_1} B_1 \arrow[r] \arrow[d, "{\langle H,L\rangle}"'] & A_1 \arrow[d, "H"] \\
A_2 \xrightarrow[F_2]{} B_2 \arrow[r] & A_2
\end{tikzcd}
$$

由此可见Adi, Nat 和Fun 是建立在Cat 之上的更广泛的三类范畴.

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Categories Fun, Nat, & Adi

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Abstract

In this paper, categories Fun, Nat and Adi are discussed, and it is proved that they are categories with products.

Difinition 1: A category is caled a factor category Fun, if

(1) Objects, all factor;

(2) Arrows, if there is a commutative diagram.

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "M"'] & B \arrow[d, "N"] \\ C \arrow[r, "G"'] & D \end{tikzcd}$$

then, we define an arrow $\langle M, N\rangle : F \to G$.

Difinition 2: If there are two commutative diagams

$$\begin{tikzcd} F \arrow[r, "m"] \arrow[d, "F_1"'] & G \arrow[d, "G_2"] \\ N \arrow[r, "n"'] & M \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "m"] \arrow[d, "G_1"'] & G \arrow[d, "F_2"] \\ N \arrow[r, "n"'] & M \end{tikzcd}$$

which mean $G_2 m = nF_1$and $F_2 m = nG_1$,we define a commutative diagram as follows:

$$\begin{tikzcd} F \arrow[r, "m"] \arrow[d, "\langle F_1, F_2\rangle"'] & G \arrow[d, "\langle G_1, G_2\rangle"] \\ M \arrow[r, "n"'] & N \end{tikzcd} \qquad (*)$$

Difinition 3: A category is called a natural transformation category Nat,if

(1) Objects, all natural transformations;

(2) Arrows, if there is a commutative diagram (*)

then,we define an arrow $\langle\langle F_1, F_2\rangle, \langle G_1, G_2\rangle\rangle : m \to n$.

Diniﬁtion 4: A category is called an adjunction category Adi, if

(1) Objects, all adjunctions;

(2) Arrows, if $\langle H, L\rangle : F_1 \to F_2$ satisfies: (a) $\langle L, H\rangle : G_1 \to G_2$; (b) there are commutative diagrams, for all objects $x \in \mathrm{Ob}X$

$$\begin{tikzcd} A_1(F_1 x_1, a_1) \arrow[r, "\varphi_1"] \arrow[d, "K"'] & X_1(x_1, G_1 a_1) \arrow[d, "L"] \\ A_2(LF_1 x_1, La_1) \arrow[d, equal] & X_2(Hx_1, HG_1 a_1) \arrow[d, equal] \\ A_2(F_2 Hx_1, La_1) \arrow[r, "\varphi_2"] & X_2(Hx_1, G_2 La_1) \end{tikzcd}$$

and $a \in \mathrm{Ob}A$. We define an arrow $\langle K, L\rangle : \langle F_1, G_1, \varphi_1\rangle \to \langle F_2', G_2', \varphi_2'\rangle$.