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A Pair of Non-trivial Combinatorial Identities*

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For the variables x and t associated with functional equation $t = x(1+x)^{-(b+1)}$, the classical Lagrange inversion formula gives the following expansion:

$$(1+x)^a = \sum_{n\geq 0} \frac{a}{a+bn}(a+bn)_n \frac{t^n}{n!}, \tag{1}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ denotes the nth rising factorial of x. A slight manipulation for equation (1) can provide another expansion:

$$\sum_{n\geq 1}(bn+1)_{n-1}\frac{t^n}{n!} = \lim_{a\to 0}\{\frac{(1+x)^a}{a} - \frac{1}{a}\} = \ln(1+x). \tag{2}$$

Based on these expressions, a combinatorial identity follows from the fact that $e^{a\ln(1+x)} = (1+x)^a$.

Theorem

$$\exp\{\sum_{n\geq 1}\frac{a}{bn}(bn)_n\frac{t^n}{n!}\} = \sum_{n\geq 0}\frac{a}{a+bn}(a+bn)_n\frac{t^n}{n!}. \tag{3}$$

By means of a simple change on the parameters involved, this identity may be reformulated as follows:

$$\exp\{\sum_{n\geq 1}\frac{a}{bn}\binom{bn}{n}t^n\} = \sum_{n\geq 0}\frac{a}{a+bn}\binom{a+bn}{n}t^n. \tag{4}$$

Replace a, b and t by aM, bM and tM^{-1} respectively in the above. Then the limiting situation of $M \to \infty$is established consequently.

Corollary

$$\exp\{\sum_{n\geq 1}\frac{a}{bn}\frac{(bn)^n}{n!}t^n\} = \sum_{n\geq 0}\frac{a}{a+bn}\frac{(a+bn)^n}{n!}t^n \tag{5}$$

For the most particular case $a = b = 1$, (5) will reduce to a beautiful combinatorial identity (cf. Comtet, p.174).

Proposition $\exp\{\sum_{n\geq 1} n^{n-1}\frac{t^n}{n!}\} = \sum_{n\geq 0}(n+1)^{n-1}\frac{t^n}{n!}$.

*L. Comtet, *Advanced Combinatorics*, D. Teidel Publ. Company, Dordrecht 1974.

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