Journal of Mathematical Research & Exposition
Vol.12, No.4, Nov. 1992

A Note on Semigroups of Linear Operators*

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It is well known [1] that a closed linear operator A with dense domain in (an ordered) Banach space E generates a C_0-semigroup if and only if the resolvent set $\rho(A)$ contains $(\omega, +\infty)$, and

$$\sup\{\| (\lambda-\omega)^n R(\lambda, A)^n \|: \lambda > \omega, n \in N\} < +\infty \tag{1}$$

for some ω. Sometimes, the condition (1) is not easy to verify. However, for the so called resolvent positive operators (that is for some $\omega, \rho(A) \supset (\omega, +\infty)$ and for all $\lambda > \omega, R(\lambda, A) = (\lambda - A)^{-1} \geq 0$), in [2], there are some easy-verified conditions. Here in this note, we establish another easy-verified condition.

Let E_+ be the positive cone of E, and

$$s(A) = \inf\{\omega : (\omega, +\infty) \subset \rho(A) \quad \text{and} \quad R(\lambda, A) \geq 0 \quad \text{for all} \quad \lambda > \omega\}, \tag{2}$$

then we have

Theorem *Let A be a densely defined resolvent positive operator, $s(A) < +\infty$. If for every $\psi \in D(A)$, there exist $\psi_1, \psi_2 \in D(A)$ such that $\psi = \psi_1 - \psi_2$, and $f_i = (\omega - A)\psi_i \in E_+$, $(i = 1, 2)$ for some $\omega > s(A)$, then A generates a positive C_0-semigroup.*

Proof Obviously, for $\lambda > \omega$

$$R(\omega, A) = R(\lambda, A) + (\lambda - \omega) R(\lambda, A) R(\omega, A). \tag{3}$$

Iterating this equation for all positive integers n yields

$$\begin{aligned} R(\omega, A) &= R(\lambda, A) + (\lambda-\omega) R(\lambda, A)^2 + (\lambda-\omega)^2 R(\lambda, A)^3 \\ &+ (\lambda-\omega)^{n-1} R(\lambda, A)^n + (\lambda-\omega)^n R(\lambda, A)^n R(\omega, A). \end{aligned} \tag{4}$$

By the positivity, it follows that for all $f \in E_+$,

$$(\lambda-\omega)^n R(\lambda, A)^n R(\omega, A) f \leq R(\omega, A) f. \tag{5}$$

Hence

$$\sup\{\| (\lambda-\omega)^n R(\lambda, A)^n R(\omega, A) f \|: \lambda > \omega, n \in N\} < +\infty. \tag{6}$$

*Received Sep. 21, 1990.

Now, for every $\psi \in D(A)$, there exist $f_1, f_2 \in E_+$ such that $\psi = R(\omega, A)(f_1 - f_2)$, thus

$$\begin{aligned} \| (\lambda-\omega)^n R(\lambda,A)^n \psi \| &= \| (\lambda-\omega)^n R(\lambda,A)^n R(\omega,A)(f_1-f_2) \| \\ &\leq (\lambda-\omega)^n R(\lambda,A)^n R(\omega,A) f_1 \| + \| (\lambda-\omega)^n R(\lambda,A)^n R(\omega,A) f_2 \| \end{aligned}$$

and

$$\sup\{\| (\lambda-\omega)^n R(\lambda,A)^n \psi \| : \lambda > \omega, n \in N\} < +\infty. \tag{7}$$

Since $D(A)$ is dense in E, (7) holds for all $\psi \in E$. By the uniform boundedness principle

$$\sup\{\| (\lambda-\omega)^n R(\lambda,A)^n \| : \lambda > \omega, n \in N\} < +\infty. \tag{8}$$

This completes the proof.

Example Let $E = L^p[a,b], 1 \leq p < +\infty$. Define $A\psi = -d\psi/dx$ with domain $D(A) = \{\psi \in E : A\psi \in E, \psi(a) = 0\}$. It is easy to known that A is densely defined, $[0, +\infty) \subset \rho(A)$, and

$$(\lambda - A)^{-1} f = \int_a^x f(t) \exp(-\lambda(x-t)) dt, \quad (\lambda \geq 0).$$

This shows that A is resolvent positive.

For each $\psi \in D(A), \psi$ is absolutely continuous, so there are [3] $\psi_i \in D(A)$ such that $\psi = \psi_1 - \psi_2$ and $-A\psi_i \in E_+$, $(i = 1, 2)$. By the Theorem, A generates a positive C_0-semigroup.

References

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