Journal of Mathematical Research & Exposition,
Vol.2, No.1, November 1993

Absolute Continuity of Vector Valued Functions*

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The definition of absolute continuity was introduced by Vitali in [5] in which he established the classical relationship between absolute continuity and the integral. Due to this relationship with the integral, much of the work with absolute continuity has dealt with finctions of intervals [4] and additive set functions [2]. Even so, there has been some interest in extending the classical notion of absolute continuity to abstract spaces. Ma [3] introduced the notions of absolute continuity and weak absolute continuity for functions defined on an interval of real numbers into a normed vector space, and Alexiewicz [1] introduced the notion of strong absolute continuity for the same type of functions.

In this paper, we consider definitions which extend strong absolute continuity and absolute continuity to functions in arbitrary topological vector spaces and weak absolute continuity to functions in locally convex spaces. We give characterizations of the functios which satisfy each of these definitions and we consider the relationship between the definitions. With these defintions we generalize some of the results in [3].

We use the following natation. The symbol I denotes a closed interval of real numbers and F denotes either the real or complex numbers. Unless stated to the contrary, a space X is understood to be a topological vector space over F whose topology is Hausdorff. If G is a subset of either X or X', G^0 denotes the polar of G taken with respect to the duality $\langle X, X'\rangle$. A subfamily $\mathcal{Y}$ of a family $\mathcal{F}$ of sets is called fundamental (in $\mathcal{F}$) if every member of $\mathcal{F}$ is contained in a suitable member of $\mathcal{Y}$. We abbreviate "neighborhood of 0" as "0−nbhd".

2. Strong Absolute Continuity

For convenience $S(R^+)$ is used to denote the collection of all finite subsets $\{\lambda_i\}$ of positive real numbers for which $\sum \lambda_i \leq 1$.

Definition 2.1 *A function f defined on I into X is said to be strongly absolutely continuous if to each 0−nbhd U in X there corresponds $\delta > 0$ such that for each finite collection $\{(x_i, y_i)\}$ of disjoint open subintervals of I with $\sum(y_i - x_i) < \delta$ there exists $\{\lambda_i\} \in S(R^+)$ such that for each i, $f(y_i) - f(x_i) \in \lambda_i U$.*

Definition 2.2 *A family $\mathcal{Y}$ of functions defined on I into X is called equi-strongly absolutely continuous if for each 0−nbhd U in X there corresponds a $\delta > 0$ such that if*

*Received Dec. 25, 1990.

The proof of the following theorem is analogous to the proof of 2.3 and so is stated without proof.

Theorem 3.3 *Let f be defined on I into a locally convex space X. If there exists a fundamental family $\mathcal{Y}$ of equi-continuous subsets of X' such that for each $G \in \mathcal{Y}$ there exists a weakly dense subset D of G for which $\{gf : g \in D\}$ is equi-absolutely continuous, then f is absolutely continuous.*

Theorem 3 of [3] follows as a corollary to this result.

Corollary 3.4 *Let f be defined on I into a normed space X and let $B = \{g \in X' :\| g \|\leq 1\}$. If f is absolutely continuous, then $\{gf : g \in B\}$ is equi-absolutely continuous. Conversely, if there exists a weakly dense subset D of B such that $\{gf : g \in D\}$ is equi-absolutely continuous, then f is absolutely continuous.*

We conclude this section with an example which shows that an absolutely continuous function need not be strongly absolutely continuous.

Example 3.5 For each $n = 1, 2, \cdots$, define $g_n : I \to R$ by $g_n(x) = \sqrt{x - (1/n)}, (1/n) \leq x \leq 1$, and $g_n(x) = 0$ otherwise. If $\mathcal{Y} = \{g_n\}$, we assert that $\mathcal{Y}$ is equi-absolutely continuous but not equi-strongly absolutely continuous.

We first show that $\mathcal{Y}$ is equi-absolutely continuous. Let $\epsilon > 0$ be given. Since $f(x) = \sqrt{x}$ is absolutely continuous on I, there exists $\delta > 0$ such that for each finite collection $\{(x_i, y_i)\}$ of disjoint open subintervals of I with $\sum(y_i - x_i) < \delta$, we have $|\sum(f(y_i) - f(x_i))| < \epsilon$. We must also have for each n, $|\sum(g_n(y_i) - g_n(x_i))| < \epsilon$; for suppose that there exists a k and a finite collection $\{(u_i, v_i)\}$ of disjoint open subintervals of I such that $\sum(v_i - u_i) < \delta$ while $|\sum(g_k(v_i) - g_k(u_i))| \geq \epsilon$. We may suppose that each $u_i \geq (1/k)$. If we let $t_i = v_i - (1/k)$ and $s_i = u_i - (1/k)$, then $\{(s_i, t_i)\}$ is a finite collection of disjoint open subintervals of I such that $\sum(t_i - s_i) < \delta$. Since $f(s_i) = g_k(u_i)$ and $f(t_i) = g_k(v_i)$ for each i, we have for this collection $|\sum(f(t_i) - f(s_i))| \geq \epsilon$, which is contrary to the choice of δ. Thus, $\mathcal{Y}$ is equi-absolutely continuous on I.

Let us now suppose that $\mathcal{Y}$ is equi-strongly absolutely continuous. Let $\epsilon > 0$ be given and let $\delta, 0 < \delta < (1/2)$, correspond to $U = (-\epsilon, \epsilon)$ in the definition of the equi-strong absolute continuity of $\mathcal{Y}$ on I. Select m so that $\delta \sum_{i=1}^{m}(1/(i+1)) > \epsilon$. For each $i = 1, 2, \cdots, m$, set $x_i = (1/(i+1)) - (\delta/(i+1))^2$ and $y_i = (1/(i+1)) + (\delta/(i+1))^2$. Then $\{(x_i, y_i)\}$ is a collection of disjoint open subintervals of I for which $\sum(y_i - x_i) < \delta$. Therefore, there must exist a set $\{\lambda_j\} \in S(R^+)$ such that for each $n, g_n(y_i) - g_n(x_i) \in \lambda_i U$. Thus, $(\delta/(i+1)) = g_{i+1}(y_i) - g_{i+1}(x_i) \in \lambda_i U$, which implies that $\delta/(i+1) \leq \lambda_i \epsilon$. We must then have $\delta \sum_{i=1}^{m}(1/(i+1)) \leq \epsilon$, which is impossible. Hence, $\mathcal{P}$ is not equi-strongly absolutely continuous on I.

Now, let c denote the space of convergent sequences of real numbers, and define $g : I \to c$ by $x \to \langle g_n(x) \rangle$. For each n, let $f_n \in c' = l_1$ be the map defined by $f_n(\langle y_k \rangle) = y_n$; then $f_n g = g_n$ for each n. Since $\{f_n g\}$ is not equi-strongly absolutely continuous, it follows from 2.4 that g is not strongly absolutely continuous. On the other hand, g is absolutely continuous. To see this, let D denote the collection of all finite sums $\sum_i \lambda_i f_{r_i}$ for which $\sum |\lambda_i| \leq 1$. D is weakly dense (in fact, strongly dense) in the unit ball of l_1. Let $\epsilon > 0$ be given and use the equi- absolute continuity of $\mathcal{Y}$ to find $\delta > 0$ which corresponds to

$U = (-\epsilon, \epsilon)$. Given $h = \sum_i \lambda_i f_{n_i} \in D$, then for each finite collection $\{(x_j, y_j)\}$ of disjoint open subintervals of I such that $\sum_j (y_j - x_j) < \delta$, we have

$$|\sum_i (hg(y_i) - hg(x_i))| = |\sum_i \sum_j \lambda_j f_{n_j}(g(y_i) - g(x_i))|$$
$$\leq \sum_j |\lambda_j| |\sum_i f_{n_j}(g(y_i) - g(x_i))| < \sum_j |\lambda_j| \epsilon \leq \epsilon.$$

Hence, $\{hg : h \in D\}$ is equi-absolutely continuous. We may now use 3.4 to conclude that g is absolutely continuous.

4. Weak Absolute Continuity

Definition 4.1 *A function f defined on I into a locally convex space X is called weakly absolutely continuous if for each $h \in X'$, hf is absolutely continuous.*

It is immediate that every absolutely continuous function is also weakly absolutely continuous. An example is given in [3] which shows that a weakly absolutely function need not be absolutely continuous.

Our next result extends Theorem 1 of [3] to functions in locally convex spaces.

Theorem 4.2 *Let X be a locally convex space and let f be defined on I into X. Let H be an absorbing subset of X', and let G denote the collection of all sums $\sum(f(y_i) - f(x_i))$ such that $\{(x_i, y_i)\}$ is a finite collection of disjoint open subintervals of I for which $\sum(y_i - x_i) < 1$. Then f is weakly absolutely continuous if and only if:*

(i) *G is a weakly bounded subset of X, and*

(ii) *These exists a strongly dense subset D of H such that for each $h \in D$, hf is absolutely continuous on I.*

Proof If f is weakly absolutely continuous, then (ii) is evidently necessary. To show that (i) is also necessary, let $h \in X'$ and let $\delta > 0$ correspond to 1 in the defintion of the absolute continuity of hf. Let m_0 be a positive integer for which $m_0 \geq (4/\delta)$. Then for each finite collection of disjoint open subintervals $\{(x_i, y_i)\}$ of I such that $\sum(y_i - x_i) < 1$, there exist m collections of disjoint open subintervals $\{(u_{i_j}, v_{i_j}) : 1 \leq j \leq n_i\}$ of I with the following properties:

(1) $m \leq m_0$.

(2) For each $i, \sum_j (v_{i_j} - u_{i_j}) < \delta$.

(3) $\sum_i \sum_j (f(v_{i_j}) - f(u_{i_j})) = \sum_k (f(y_k) - f(x_k))$. Thus, for any finite collection of disjoint open subintervals $\{(x_i, y_i)\}$ of I with $\sum(y_i - x_i) < 1$, we have

$$|h(\sum_k (f(y_k) - f(x_k))| \leq \sum_i |h(\sum_j (f(v_{i_j}) - f(u_{i_j}))| \leq m \leq m_0.$$

Hence, $\sup\{h(z)| : z \in G\} \leq m_0$, and, since h is arbitrary, it follows that G is weakly bounded.

Conversely, suppose the condition is satisfied and let K denote the circled hull of G. Then, given $h \in H$ and $\epsilon > 0$, there exists $g \in (h + (\epsilon/2K^0) \cap D$. Since $g \in D$, there exists $\delta, 0 < \delta < 1$, which corresponds to $\epsilon/2$ in the definition of the absolute continuity of gf. If $\{(x_i, y_i)\}$ is a finite collection of disjoint subintervals of I such that $\sum(y_i - x_i) < \delta$, then

$$\begin{aligned} &|\sum h(f(y_i) - f(x_i))| \\ &\leq |(h-g)(\sum f(y_i) - f(x_i))| + |\sum g(f(y_i) - f(x_i))| \\ &< (\epsilon/2) + (\epsilon/2) = \epsilon. \end{aligned}$$

Thus, hf is absolutely continuous. Since H is absorbing, we can conclude that gf is absolutely continuous for each $g \in X'$ which completes the proof.

I want to thank Professor Yi-Chun Zhao.

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向量值函数的绝对连续性

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摘　　要

本文将强绝对连续性和绝对连续性两个概念推广到取值于任意拓扑向量空间的函数,和将弱绝对连续性推广到取值于局部凸空间的函数.描述了这些概念之间的关系及特征,并推广了马绍群[3]的一些结果.