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Global Solution for Degenerate Nonlinear Reaction-Diffusion System*

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Abstract In this paper, we investigate the global exstence and nonexistence of solution for degenerate nonlinear parabolic system

$$u_{it} = \Delta\eta_i(u_i) + f_i(x,t,u_1,\cdots,u_N), \quad (x,t)\in Q_T = \Omega\times(0,T),$$

with Dirichlet boundary conditions.

1. Introduction

We consider a system of degenerate nonlinear parabolic equations for $i = 1,2,\ldots,N$,

$$u_{it} = \Delta\eta_i(u_i) + f_i(x,t,u_1,\cdots,u_N), \quad (x,t)\in Q_T = \Omega x(0,T), \tag{1.1}$$

here $\eta_i(u_i), i = 1,2,\ldots,N$ are nonnegative smooth functions, the model of this type is the porous medium equation. Population dynamics also arises this type model in which $\eta_i(u_i) = |u_i|^{m_i}\operatorname{sign}(u_i)$.

The degenerate nonlinear reaction-diffusion equations have been given extensive attention by many authors in recent year. For a single equation $N = 1$, Aronson, Crandall and Peletier discussed the solvability and asymptotic behavior of solution of the inital and Dirichlet problem for (1.1) with $\eta(u) = |u|^m\operatorname{sign}(u)$ in $n = 1$. Bortsch [4] discussed the existence and stabilization of solution for (1.1) in n-dimensional case. Sacks [7], Livin and Sacks [8] studied the global existence and nonexistence of solution for (1.1). Maddalena [9] proved the existence of global solution for a reaction- diffusion system

$$u_{it} = \Delta u_i^{m_i} + f_i(u), \tag{1.2}$$

with Dirichlet boundary conditions, $m_i > 1$. Wang Yuanming [5] studied the global existence and nonexistence of solution for a more general system (1.1) $N > 1$. But concerning systems require that the nonlinearlies be quasimonotone nondecreasing or nonincreasing in $u_1, u_2, \cdots, u_N$. However, there are many examples from applications where this restriction is not satisfied.

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The outlin of this paper is as follows. In section 2, we state some assumptions and the fundamental definitions and establish a monotone iteration scheme for the solution of the Dirichlet problem of (1.1). In section 3, we prove existence comparison theorem. In section 4, we extend the main results in [5] and revise the proof of theorem 2 in it.

2. Uniqueness and comparison Theorem

We consider the initial boundary value problem

$$u_{it} = \Delta\eta_i(u_i) + f_i(x,t,u_1,u_2,\cdots,u_N),\quad (x,t)\in Q_T=\Omega\times(0,T), \tag{2.1}$$

$$u_i(x,t) = g_i(x,t),\quad (x,t)\in\Gamma_T=\partial\Omega\times(0,T), \tag{2.2}$$

$$u_i(x,t)|_{t=0} = \psi_i(x),\quad x\in\Omega, i=1,2,\ldots,N, \tag{2.3}$$

where Ω is a bounded domain in $\mathbf{R^n}$ with smooth boundary $\partial\Omega, u=(u_1,u_2,\cdots,u_N): Q_T\to\mathbf{R^N}, x=(x_1,x_2,\cdots,x_n)\in\mathbf{R^n}, n\geq 1$.

We introduce the notation $f_i(x,t,u_i,V_i,W_i)$, where $V_i=(u_{1_i},u_{2_i},\cdots,u_{\beta_i}), W_i=(w_{1'_i}, w_{2'_i},\cdots,w_{k'_i})(k_i+k'_i=N-1)$. We assume that $f_i(x,t,u_i,V_i,W_i)=f(x,t,u_1,u_2,\cdots,u_N)$ is quasimonotone nondecreasing in u_i and V_i, and monotone nonincreasing in W_i. Suppose that the functions $\eta(u_i), f_i(x,t,u_i,V_i,W_i)$ and $\psi_i(x)$ satisfy the following set of hypothesis:

(H$_1$) $\eta_i\in C^1([0,\infty))\cap C^3((0,\infty)), \eta_i(0)=0, \eta'_i(s)\geq\varepsilon_i(\delta)$ for $s\geq\delta, \eta'_i(s)\leq\mu_i(M)$ for $s\leq M$, here $\delta>0, M>0$ and $\varepsilon_i(\cdot),\mu_i(\cdot)$ are positive functions in $(0,\infty)$.

(H$_2$) $f_i(x,t,u_i,V_i,W_i)(i=1,2,\ldots,N)$ satisfy the above monotone properties, $f_i(x,t,u_i,V_i,W_i)$ are Lipschitz continuous on $\Omega\times[0,T]\times[-M,M]^N$, there exists positive constants C_i,C'_i,C''_i such that

$$\begin{aligned}&-C_i(u_i-\tilde{u}_i)-C'_i\sum_{j_i}(u_{j_i}-\tilde{u}_{j_i})+C''_i\sum_{j'_i}(u_{j'_i}-\tilde{u}_{j'_i})\\ &\leq f_i(x,t,u_i,V_i,W_i)-f_i(x,t,\tilde{u}_i,\tilde{V}_i,\tilde{W}_i)\\ &\leq C_i(u_i-\tilde{u}_i)+C'_i\sum_{j_i}(u_{j_i}-\tilde{u}_{j_i})-C''_i\sum_{j'_i}(u_{j'_i}-\tilde{u}_{j'_i})\end{aligned} \tag{2.4}$$

for any $u_j,\tilde{u}_j\in[-M,M], u_j\geq\tilde{u}_j, j=1,2,\ldots,N$.

(H$_3$) $\psi_i(x)\in L^\infty(\Omega), i=1,2,\ldots,N$.

Definition 1 *A nonnegative function set $u=(u_1,u_2,\cdots,u_N)$ is said to be a weak solution of problem (2.1)-(2.3) in $Q_T=\Omega\times(0,T), T>0$, if*

(i) $u_i\in C([0,T];L^1(\Omega))\cap L^\infty(Q_T)$;

(ii) u_i *satisfies the identity*

$$\int_\Omega\varphi_i(x,T)u_i(x,T)dx=\int_{Q_T}[\varphi_i f_i(x,t,u_i,V_i,W_i)+\eta_i(u_i)\Delta\varphi_i+\varphi_{it}u_i]dxdt$$

$$+\int_\Omega \varphi_i(x,0)\psi_i(x)dx - \int_{\Gamma_T} \eta_i(g_i)\frac{\partial \varphi_i}{\partial n}dsdt \tag{2.5}$$

for every $\varphi_i(x,t) \in C^{1,0}(\overline{Q}_T) \cap C^{2,1}(Q_T)$ *such that* $\varphi_i = 0$ *on* $\Gamma_T = \partial\Omega \times (0,T]$;

(iii) $u_i(x,t) = g_i(x,t), (x,t) \in \Gamma_T$ *is the sense of the traces,* $i = 1,2,\ldots,N$. *Here* $\frac{\partial}{\partial n}$ *is the outward normal derivative on* Γ_T.

Definition 2 Let nonnegative vector function $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_N)$ and $\underset{\sim}{U} = (\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_N)$ satisfy the assumption (i), (iii) in Definition 1, $\tilde{U} \geq \underset{\sim}{U}$. $\tilde{U}, \underset{\sim}{U}$ are called upper and lower solutions of problem (2.1)-(2.3), if the following inequalities hold:

$$\int_\Omega \varphi_i(x,T)\tilde{u}_i(x,T)dx \geq \int_{Q_T} [\varphi_i f_i(x,t,\tilde{u}_i,\tilde{V}_i,\underset{\sim}{W}_i) + \eta_i(\tilde{u}_i)\Delta\varphi_i + \varphi_{it}\tilde{u}_i]dxdt$$

$$+\int_\Omega \varphi_i(x,0)\tilde{u}_i^0(x)dx - \int_{\Gamma_T} \eta_i(g_i)\frac{\partial \varphi_i}{\partial n}dsdt; \tag{2.6}$$

$$\int_\Omega \varphi_i(x,T)\underset{\sim}{u}_i(x,T)dx \leq [\int_{Q_T} \varphi_i f_i(x,t,\underset{\sim}{u}_i,\underset{\sim}{V}_i,\tilde{W}_i) + \eta_i(\underset{\sim}{u}_i)\Delta\varphi_i + \varphi_{it}\underset{\sim}{u}_i]dxdt$$

$$+\int_\Omega \varphi_i(x,0)\underset{\sim}{u}_i^0(x)dx - \int_{\Gamma_T} \eta_i(g_i)\frac{\partial \varphi_i}{\partial n}dsdt \tag{2.7}$$

for every nonnegative function $\varphi_i(x,t) \in C^{1,0}(\overline{Q}_T) \cap C^{2,1}(Q_T)$ such that $\varphi_i = 0$ on Γ_T, where $\tilde{u}_i(x,t)|_{t=0} = \tilde{u}_i^0(x) \geq \psi_i(x) \geq \underset{\sim}{u}_i(x) = \underset{\sim}{u}_i(x,t)|_{t=0}, i = 1,2,\ldots,N$.

For the application in tater we need the following basic lemma.

Lemma 1 *Let* $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_N)$ *and* $\underset{\sim}{U} = (\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_N)$ *be upper and lower solutions of problem (2.1)-(2.3) with the data* $\tilde{u}_i^0(z), \tilde{g}_i(x,t)$ *and* $\underset{\sim}{u}_i^0(x), \underset{\sim}{g}_i(x,t)$ *respectively,* $\tilde{u}_i^0(x) \geq u$ $\tilde{u}_i^0(x), \tilde{g}_i(x,t) \geq \underset{\sim}{g}_i(x,t)$, *then for* $0 < t \leq T$ *and each* $\lambda > 0$.

$$e^{\lambda t}\int_\Omega (\underset{\sim}{u}_i(x,t) - \tilde{u}_i(x,t))^+ dx \leq \int_\Omega (\underset{\sim}{u}_i^0(x) - \tilde{u}_i^0(x))^+ dx$$

$$+\int_{Q_t} e^{\lambda\tau}[f_i(x,\tau,\underset{\sim}{u}_i,\underset{\sim}{V}_i,\tilde{W}_i) - f_i(x,\tau,\tilde{u}_i,\tilde{V}_i,\underset{\sim}{W}_i)$$

$$+\lambda(\underset{\sim}{u}_i(x,\tau) - \tilde{u}_i(x,\tau))]^+ dxd\tau, \tag{2.8}$$

$i = 1,2,\ldots,N$, where $r^+ = \max(r,0), Q_t = \Omega \times (0,t)$.

This lemma has been proved in the case $n = 1, N = 1$ in [3]. The proof for the case $n > 1, N > 1$ has not essential difficults and we omit it here.

Lemma 2 *Let* $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_N)$ *and* $\underset{\sim}{U} = (\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_N)$ *be upper and lower solutions of problem (2.1)-(2.3), then*

$$\underset{\sim}{u}_i(x,t) \leq \tilde{u}_i(x,t) \quad (x,t) \in Q_T, i = 1,2,\ldots,N. \tag{2.9}$$

Proof By the monotone properties of $f_i(x,t,u_i,V_i,W_i)$ there exists λ_i such that

$$\lambda_i u_i + f_i(x,t,u_i,V_i,W_i)$$

is quasimonotone nondecreasing in u_i and V_i and monotone noncreasing in W_i. Suppose that

$$\lambda_i(\underset{\sim}{u}_i(x,t)-\tilde{u}_i(x,t))+f_i(x,t,\underset{\sim}{u}_i,\underset{\sim}{V}_i,\tilde{W}_i)-f_i(x,t,\tilde{u}_i,\tilde{V}_i,\underset{\sim}{W}_i)>0,$$

that is

$$\begin{aligned}\lambda_i(\underset{\sim}{u}_i(x,t)-\tilde{u}_i(x,t))&+[f_i(x,t,\underset{\sim}{u}_i,\underset{\sim}{V}_i,\tilde{W}_i)-f_i(x,t,\tilde{u}_i,\underset{\sim}{V}_i,\tilde{W}_i)\\&+\sum_{j_i}[f_i(x,t,\tilde{u}_i,\cdots,\underset{\sim}{u}_{j_i},\cdots,\tilde{W}_i)-f_i(x,t,\tilde{u}_i,\cdots,\tilde{u}_{j_i},\cdots,\tilde{W}_i)\\&+\sum_{j_i'}[f_i(x,t,\tilde{u}_i,\tilde{V}_i,\cdots,\tilde{u}_{j_i'},\cdots)-f_i(x,t,\tilde{u}_i,\tilde{V}_i,\cdots,\underset{\sim}{u}_{j_i'},\cdots,)]>0,\end{aligned}$$

here the quantity in the bracket represent the variation of function f_i which follows from the variation of one component u_{j_i} (or $u_{j_i'}$) only. From above inequality and the momotone properties of f_i it's easy to see that

$$\underset{\sim}{u}_i(x,t)\geq\tilde{u}_i(x,t)\quad(x,t)\in Q_T,i=1,2,\ldots,N. \tag{2.10}$$

By the condition (2.4) and (2.10) yield

$$\begin{aligned}e^{\lambda_i t}\int_\Omega[\underset{\sim}{u}_i(x,t)-\tilde{u}_i(x,t)]^+dx&\leq\int_\Omega[\underset{\sim}{u}_i^0(x)-\tilde{u}_i^0(x)]^+dx+\int_{Q_t}e^{\lambda_i\tau}[(C_i+\lambda_i)(\underset{\sim}{u}_i-\tilde{u}_i)\\&+C_i'\sum_{j_i}(\underset{\sim}{u}_{j_i}-\tilde{u}_{j_i})-C_i''\sum_{j_i'}(\underset{\sim}{u}_{j_i'}-\tilde{u}_{j_i'})]^+dxd\tau,\ i=1,2,\ldots,N.\end{aligned}$$

Summing above inequalities over i from 1 to N we get

$$\begin{aligned}\int_\Omega\sum_{i=1}^N e^{\lambda_i t}[\underset{\sim}{u}_i(x,t)-\tilde{u}_i(x,t)]^+dx&\leq\int_\Omega\sum_{i=1}^N[\underset{\sim}{u}_i^0(x)-\tilde{u}_i^0(x)]^+dx\\+\int_{Q_t}[\max_{1\leq i\leq N}(C_i+\lambda_i)&+\max_{1\leq i\leq N}(C_i'+C_i'')\sum_{j=1}^N e^{\lambda_j T}\sum_{i=1}^N[\underset{\sim}{u}_i(x,\tau)-\tilde{u}_i(x,\tau)]^+dxd\tau.\end{aligned}$$

The Gronwall's inequality shows that

$$\begin{aligned}\int_\Omega\sum_{i=1}^N[\underset{\sim}{u}_i(x,t)-\tilde{u}_i(x,t)]^+dx&\leq\int_\Omega\sum_{i=1}^N e^{\lambda_i t}[\underset{\sim}{u}_i(x,t)-\tilde{u}_i(x,t)]^+dx\\&\leq e^{Kt}\int_\Omega\sum_{i=1}^N[\underset{\sim}{u}_i^0(x)-\tilde{u}_i^0(x)]^+dx,\end{aligned}\tag{2.11}$$

where $K=\max_{1\leq i\leq N}(C_i+\lambda_i)+\max_{1\leq i\leq N}(C_i'+C_i'')\sum_{j=1}^N e^{\lambda_j T}$. It follows that $\underset{\sim}{u}_i(x,t)\leq\tilde{u}_i(x,t)$ which constradicts (2.10). Hence $\lambda_i(\underset{\sim}{u}_i-\tilde{u}_i)+f_i(x,t,\underset{\sim}{u}_i,\underset{\sim}{V}_i,\tilde{W}_i)-f_i(x,t,\tilde{u}_i,\tilde{V}_i,\underset{\sim}{W}_i)\leq 0$. Therefore (2.8) implies that $\underset{\sim}{u}_i(x,t)\leq\tilde{u}_i(x,t),i=1,2,\ldots,N$.

From Lemma 2 we immediately obtain the following corollary.

Corollary 1 *Let $u = (u_1, u_2, \cdots, u_N)$ and $\hat{u} = (\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_N)$ be the solutions of problem (2.1)-(2.3) with the inital data $\psi_i(x)$ and $\hat{\psi}_i(x)$ respectively. Then for $g_i(x,t) = 0$, we have*

$$\sum_{i=1}^{N} \|u_i(\ \ ,t) - \hat{u}_i(\ \ ,t)\|_{L'(\Omega)} \le e^{Kt} \sum_{i=1}^{N} \|\psi_i - \hat{\psi}_i\|_{L^1(\Omega)},$$

where $K = N \max\{\max_{1\le i\le N} C_i, \max_{1\le i\le N} C_i', \max_{1\le i\le N} C_i''\}$.

Corollary 2 *Problem (2.1)-(2.3) ($g(x,t) = 0$) has only one bounded weak solution in $Q_T, T > 0$.*

In order to establish an existence comparison theorem in terms of upper and lower solutions. We choose upper solution $\tilde{U} = (\tilde{u}_i, \tilde{u}_i, \cdots, \tilde{u}_N)$ and lower solution $\underset{\sim}{U} = (\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_N)$ as two distinct initial iteration to construct two sequences $\{\overline{U}^{(k)}\} = \{(\bar{u}_1^{(k)}, \bar{u}_2^{(k)}, \cdots, \bar{u}_N^{(k)})\}$ and $\{\underline{U}^{(k)}\} = \{(\underline{u}_1^{(k)}, \underline{u}_2^{(k)}, \cdots, \underline{u}_N^{(k)},)$ which obtain from the identity

$$\int_\Omega \varphi_i(x,T)\bar{u}_i^{(k)}(x,T)dx - \int_{Q_T} [\eta(\bar{u}_i^{(k)})\Delta\varphi_i + \varphi_i + \varphi_{i_t}\bar{u}_i^{(k)}]dxdt - \int_\Omega \varphi_i(x,0)\psi_i(x)dx$$

$$= \int_{Q_T} \varphi_i f_i(x,t,\bar{u}_i^{(k-1)}, \overline{V}_i^{(k-1)}, \underline{W}_i^{(k-1)})dxdt, \tag{2.12}$$

and

$$\int_\Omega \varphi_i(x,T)\underline{u}_i^{(k)}(x,T)dx - \int_{Q_T} [\eta(\underline{u}_i^{(k)})\Delta\varphi_i + \varphi_i + \varphi_{i_t}\underline{u}_i^{(k)}]dxdt - \int_\Omega \varphi_i(x,0)\psi_i(x)dx$$

$$= \int_{Q_T} \varphi_i f_i(x,t,\underline{u}_i^{(k-1)}, \underline{V}_i^{(k-1)}, \overline{W}_i^{(k-1)})dxdt, \tag{2.13}$$

$k = 1, 2, \ldots$. The existence of $\bar{u}_i^{(k)}$ and $\underline{u}_i^{(k)}$ can be derived from paper [2].

Lemma 3 *Let the hypothesis (H_1)-(H_3) hold and let $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_N)$ and $\underset{\sim}{U} = (\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_N)$ be a pair weak upper and lower solutions of problem (2.1)-(2.3). Then the maximal sequence $\{\overline{U}^{(k)}\} = \{(\bar{u}_1^{(k)}, \bar{u}_2^{(k)}, \cdots, \bar{u}_N^{(k)})$ is monotone nonincreasing and the minimal sequence $\{\underline{U}^{(k)}\} = \{(\underline{u}_1^{(k)}, \underline{u}_2^{(k)}, \cdots, \underline{u}_N^{(k)})$ is monotone nondecreasing. Moreover*

$$0 \le \underset{\sim}{u}_i \le \underline{u}_i^{(1)} \le \cdots \le \underline{u}_i^{(k)} \le u_i^{(k+1)} \le \cdots \le \bar{u}_i^{(k+1)} \le \bar{u}_i^{(k)} \le \cdots \le \bar{u}_i^{(1)} \le \tilde{u}_i, \tag{2.14}$$

$i = 1, 2, \ldots, N$.

Proof from (2.7),(2.13) and inequality (2.8) ($\lambda = 0$) we obtain

$$\int_\Omega [\underset{\sim}{u}_i(x,t) - \underline{u}_i^{(1)}(x,t)]^+ dx \le \int_\Omega [\psi_i(x) - \psi_i(x)]^+ dx$$

$$\int_{Q_t} [f_i(x,\tau,\underset{\sim}{u}_i, \underset{\sim}{V}_i, \tilde{W}_i) - f_i(x,\tau,\underset{\sim}{u}_i, \underset{\sim}{V}_i, \tilde{W}_i)]^+ dx = 0.$$

This implies $\underset{\sim}{u}_i(x,t) \le \underline{u}_i^{(1)}(x,t)$. Similarly, we prove $\bar{u}_i^{(1)}(x,t) \le \tilde{u}_i(x,t)$. We show out

$\underline{u}_i^{(1)}(x,t) \leq \bar{u}_i^{(1)}(x,t)$. It follows from (2.12), (2.13) ($k=1$) and (2.8) ($\lambda=0$) that

$$\int_\Omega [\underline{u}_i^{(1)}(x,t) - \bar{u}_i^{(1)}(x,t)]^+ dx \leq \int_{Q_t} [f_i(x,\tau,\underset{\sim}{u}_i,\underset{\sim}{V}_i,\tilde{W}_i) - f_i(x,\tau,\tilde{u}_i,\tilde{V}_i,\underset{\sim}{W}_i,)]dxd\tau$$

$$\leq \int_{Q_t} [C_i(\underset{\sim}{u}_i - \tilde{u}_i) + C_i'\sum_{j_i}(\underset{\sim}{u}_{j_i} - \tilde{u}_{j_i}) - C_i''\sum_{j_i'}(\tilde{u}_{j_i'} - \underset{\sim}{u}_{j_i'})]dxd\tau = 0,$$

here we made used of the conclusion of Lemma 2; $\tilde{u}_i \geq \underset{\sim}{u}_i$. From this $\underline{u}_i^{(1)}(x,t) \leq \bar{u}_i^{(1)}(x,t)$. Hence

$$\underset{\sim}{u}_i(x,t) \leq \underline{u}_i^{(1)}(x,t) \leq \bar{u}_i^{(1)}(x,t) \leq \tilde{u}_i(x,t),\ i=1,2,\ldots,N. \tag{2.15}$$

We assume, by induction, that $\underline{u}_i^{(k-1)} \leq \underline{u}_i^{(k)}, \bar{u}_i^{(k)} \leq \bar{u}_i^{(k-1)}, \underline{u}_i^{(k)} \leq \bar{u}_i^{(k)}, i=1,2,\ldots,N$, $k=1,2,\ldots,k_0$. By the same argument as in the proof of the relation $\underline{u}_i^{(1)} \leq \bar{u}_i^{(1)}$ we get

$$\int_\Omega [\underline{u}_i^{k_0+1}(x,t) - \bar{u}_i^{(k_0+1)}(x,t)]^+ dx$$

$$\leq \int_{Q_t} [f_i(x,\tau,\underline{u}_i^{(k_0)},\underline{V}_i^{(k_0)},\overline{W}_i^{(k_0)}) - f_i(x,\tau,\bar{u}_i^{(k_0)},\overline{V}_i^{(k_0)},\underline{W}_i^{(k_0)},)]dxd\tau$$

$$\leq \int_{Q_t} [C_i(\underline{u}_i^{(k_0)} - \bar{u}_i^{(k_0)}) + C_i'\sum_{j_i}(\underline{u}_{j_i}^{(k_0)} - \bar{u}_{j_i}^{(k_0)}) - C_i''\sum_{j_i'}(\bar{u}_{j_i'}^{(k_0)} - \underline{u}_{j_i'}^{(k_0)})]dxd\tau = 0.$$

It follows $\underline{u}_i^{(k_0+1)} \leq \bar{u}_i^{(k_0+1)}$. Similarly $\underline{u}_i^{(k_0)} \leq \underline{u}_i^{(k_0+1)}, \bar{u}_i^{(k_0)} \leq \bar{u}_i^{(k_0+1)}, i=1,2,\ldots,N$. This completes the proof of lemma 3.

3. Existence-Comparison Theorem

We now prove the existence-comparison theorem as the following

Theorem 1 *Let $\tilde{U}=(\tilde{u}_1,\tilde{u}_2,\cdots,\tilde{u}_N)$ and $\underset{\sim}{U}=(\underset{\sim}{u}_1,\underset{\sim}{u}_2,\cdots,\underset{\sim}{u}_N)$ be a pair upper and lower solutions of problem (2.1)-(2.3). Suppose that the hypothesis (H_1)-(H_3) hold, $\psi_i(x) \geq 0$, Then the maximal sequence $\{\overline{U}^{(k)}\} = \{(\bar{u}_1^{(k)},\bar{u}_2^{(k)},\cdots,\bar{u}_N^{(k)})\}$ and minimal sequence $\{\underline{U}^{(k)}\} = \{(\underline{u}_1^{(k)},\underline{u}_2^{(k)},\cdots,\underline{u}_N^{(k)})\}$ converge monotone from above and below, respectively, to a unique solution of problem (2.1)-(2.3).*

Proof In view of Lemma 3 the pointwise limit

$$\lim_{k\to\infty} \bar{u}_i^{(k)}(x,t) = \bar{u}_i(x,t),\quad \lim_{k\to\infty} \underline{u}_i^{(k)}(x,t) = \underline{u}_i(x,t) \text{ a.e. } (x,t)\in Q_T, i=1,2,\cdots,N \tag{3.1}$$

exist and $\underset{\sim}{u}_i(x,t) \leq \underline{u}_i(x,t) \leq \bar{u}_i(x,t) \leq \tilde{u}_i(x,t)$, a.e., on $Q_T, i=1,2,\ldots,N$. Our main objective is to prove that $(\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_N) = (\underline{u}_1,\underline{u}_2,\cdots,\underline{u}_N)$ and it is an unique solution of problem (2.1)-(2.3). For this purpose, letting $k\to\infty$ in the identity (2.12) and (2,13) the limit function set $(\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_N) = (\underline{u}_1,\underline{u}_2,\cdots,\underline{u}_N)$ satisfy the following integral identity

$$\int_\Omega \varphi_i(x,T)\bar{u}_i(x,T)dx = \int_{Q_T}[\varphi_i f_i(x,t,\bar{u}_i,\overline{V}_i,\underline{W})$$

$$+\eta_i(\bar{u}_i)\Delta\varphi_i + \varphi_{i_t}\bar{u}_i]dxdt + \int_\Omega \varphi_i(x,0)\psi_i(x)dx, \tag{3.2}$$

and

$$\int_\Omega \varphi_i(x,T)\underline{u}_i(x,T)dx = \int_\Omega[\varphi_i f_i(x,t,\underline{u}_i,\underline{V}_i,\overline{W})$$

$$+\eta_i(\underline{u}_i)\Delta\varphi_i + \varphi_{i_t}\underline{u}_i]dxdt + \int_\Omega \varphi_i(x,0)\psi_i(x)dx. \tag{3.3}$$

We consider the initial boundary value problem for $2N$ equations

$$w_{i_t} = \Delta\eta_i(w_i) + f_i^*(x,t,w_1,w_2,\cdots,w_N), \quad (x,t)\in Q_T, \tag{3.4}$$

$$w_i(x,t) = 0 \qquad (x,t)\in\Gamma_T, \tag{3.5}$$

$$w_i(x,t)|_{t=0} = \psi_i^*(x, \ ) \qquad x\in\Omega, \tag{3.6}$$

where

$$f^*(x,t,w_1,w_2,\cdots,w_{2N}) = \begin{cases} f_i(x,t,\hat{u}_i,\hat{V}_i,W_i^*) & i=1,2,\dots,N, \\ f_{N+i}(x,t,u_i^*,V_i^*,\hat{W}_i) & i=N+1,N+2,\cdots,2N, \end{cases} \tag{3.7}$$

$$\eta_i(w_i) = \eta_i(\hat{u}_i), i=1,2,\dots,N; \quad \eta_{N+i}(w_{N+i}) = \eta_i(u_i^*), i=1,2,\dots,N, \tag{3.8}$$

$$\psi_i^*(x) = \psi_{N+i}^* = \psi_i(x), i=1,2,\dots,N; \qquad w_i=\hat{u}_i, w_{N+i}=u_i^*, i=1,2,\dots,N. \tag{3.9}$$

From (3.2), (3.3) it is clear that $(w_1,w_2,\cdots,w_{2N}) = (\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_N,\underline{u}_1,\underline{u}_2,\cdots,\underline{u}_N)$ is a weak solution of problem (3.4)–(3.6). By special form of f_i^*, $(\underline{u}_1,\underline{u}_2,\cdots,\underline{u}_N)$ and $(\bar{u}_1,\bar{u}_2,\cdots,$

$\bar{u}_N)$ are weak solutions of problem (3.4)–(3.6). Evidently, Corollary 2 can apply to problem (3.4)–(3.6). By uniqueness we obtain $(\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_N) = (\underline{u}_1,\underline{u}_2,\cdots,\underline{u}_N)$. The proof of Theorem is completed.

Example (see [3],[6]).

We consider the following reaction-diffusion problem:

$$u_t = (u^m)_{xx} + u^\lambda, \qquad (x,t)\in(-L,L)\times(0,T], \tag{3.10}$$

$$u(\pm L,t) = 0, \tag{3.11}$$

$$u(x,t)|_{t=0} = u_0(x), \qquad x\in(-L,L), \tag{3.12}$$

where $u_0(x)\geq 0, u_0\neq 0, u_0(x)\in L^\infty((-L,L)), m>\lambda\geq 1$.

The usual integration method shows that any nontrivial steady solution of problem

$$0 = (u^m)_{xx} + u^\lambda, \quad x\in(-L,L) \tag{3.13}$$

$$u(\pm L) = 0 \tag{3.14}$$

is given by the formula

$$\sqrt{m(m+\lambda)}\mu^{\frac{m-\lambda}{2}}\int_{\frac{u(x)}{\mu}}^{1}\frac{r^{m-1}}{\sqrt{1-r^{m+\lambda}}}dr=\sqrt{2}|x|, \tag{3.15}$$

where $u(0)=\mu$ and

$$L=L(\mu)=\sqrt{\frac{m(m+\lambda)}{2}}\int_0^1\frac{r^{m-1}}{\sqrt{1-r^{m+\lambda}}}dr.$$

Suppose $0\le u_0(x)\le u(x)$, where $u(x)$ is a nontrivial solution of problem (3.13)–(3.14). We choose $\tilde{u}(x)$ as a solution of problem (3.13)–(3.14) in $(-L_1,L_1)\times(0,T]$, where $L_1>L$. Then it's easy to check $\tilde{u}(x)$ is an upper solution of problem (3.10)–(3.12), and $\underset{\sim}{u}(x)=0$ is a lower solution. Therem 1 asserts that problem (3.10)–(3.12) has an unique solution.

4. Global Existence and Nonexistence

In this section, we extend some results in [5].

Theorem 2 *Let the hypothesis* $(H_1)--(H_3)$ *hold. If* $\psi_i(x)\ge 0$ *and* $f_i(x,t,u_i,V_i,W_i)\ge f_i(x,t,0,0,W_i)\ge 0$ *for any* $(x,t)\in Q=\Omega\times(0,\infty), 0\le W_i\le M$ *(e.i.,* $0\le u_{j_i'}\le M, j_i'=1_i',\dots,k_i'$*) and*

$$|f_i(x,t,u_1,u_2,\cdots,u_N)|\le C_i(x,t)+\sum_{j=1}^{N}C_{ij}(x,t)[\eta_j(u_j)]^{\alpha_{i,j}}, \tag{4.1}$$

where $0<\alpha_{i,j}<1, C_i(x,t)$ *and* $C_{ij}(x,t)$ *are continuous functions on* $\overline{Q}=\overline{\Omega}\times[0,\infty)$*. Then problem (2.1)– (2.3)* $(g_i(x,t)=0)$ *has an unique weak solution on* Q_T *for any positive* T.

Proof The proof of this Theorem is similar to that of Theorem 2 in [5]. Let $(R_p)_{p\in N}$ be an increasing sequence of positive real numbers such that $\lim_{p\to\infty}R_p=\infty$ and f_{ip} be nondecreasing in u_i and V_i and nonincreasing W_i smooth functions that linearized of function f_i for $|u|=(\sum_{i=1}^{N}u_i^2)^{\frac{1}{2}}\ge R_p$ and $0\le f_{ip}\le f_i$ for $u_i\ge 0, i=1,2,\dots,N$.

To approximate the initial function $\psi_i(x)$, pick $(\psi_{ip}(x))_{p\in N}$ such that $\psi_{ip}(x)\in C_0^\infty(\Omega)$, $\psi_{ip}(x)\ge 0, \|\psi_{ip}\|_{L^\infty(\Omega)}\le\|\psi_i\|_{L^\infty(\Omega)}$ and $\psi_{ip}(x)\to\psi_i(x)$ in $L^2(\Omega)$. We consider the following regularizing problem for $p=1,2,\dots,$

$$(u_{ip})_t=\Delta\eta_i(u_{ip})+f_{ip}(x,t,u_p-\tfrac{1}{p}) \qquad (x,t)\in Q_T, \tag{4.2}$$

$$u_{ip}(x,t)=1/p, \qquad (x,t)\in\Gamma_T \tag{4.3}$$

$$u_{ip}(x,t)|_{t=0}=\psi_{ip}(x)+1/p. \qquad x\in\Omega \tag{4.4}$$

$i=1,2,\dots,N$ and give a priori estimate for the solutions, where $u_p-1/p=(u_{1p}-1/p,u_{2p}-1/p,\cdots,u_{Np}-1/p)$.

To prove Theorem we need the following Lemmas.

Lemma 4 *For every $p \geq 1$ problem (4.2)-(4.4) has an unique classical solution $u_p = (u_{1p}, u_{2p}, \cdots, u_{Np})$ and*

$$u_{ip}(x,t) \geq 1/p \quad (x,t) \in Q_T, i = 1,2,\ldots,N. \tag{4.5}$$

Lemma 5 *The following estimate hold for the solutions of problem (4.2)-(4.4)*

$$\|u_{ip}\|_{L^\infty(Q_T)} \leq C, \quad i = 1,2,\ldots,N, p = 1,2,3,\ldots, \tag{4.6}$$

where C is a constant depending on $\alpha_{ij}, n, N, |\Omega|, T, \|C_i\|_{L^\infty(Q_T)}$ and $\|C_{ij}\|_{L^\infty(Q_T)}$.

Lemma 6 *Under the assumptions of Lemma 5 and $\psi_i(x) \in L^\infty(\Omega) \cap H_0^1(\Omega)$,*

$$\|\eta_i(u_{ip})\|_{L^2(Q_T)}, \| \nabla \eta_i(u_{ip})\|_{L^\infty([0,T),L^2(\Omega))} \leq C \tag{4.7}$$

where C is a constant which depends on $\alpha_{ij}, n, N, |\Omega|, T, \|C_i\|_{L^2(Q_T)}$, and $\|C_{ij}\|_{L^\infty(Q_T)}$.

The proof of Lemma 4 and 6 are same as in the proof of Lemma 1 and 3 in [5]. We omit it here.

We now prove Lemma 5. Let

$$v_{ip} = \eta_i(u_{ip}) + \frac{1}{p} - \eta_i(\frac{1}{p}), \text{ i.e., } \beta_i(v_{ip} - \frac{1}{p} + \eta_i(\frac{1}{p})) = u_{ip},$$

where $\beta_i = \eta_i^{-1}$. Then the equation (4.2) can be rewritten as

$$\beta_i'(v_{ip} - \frac{1}{p} + \eta_i(\frac{1}{p})) = \Delta(v_{ip}) + f_{ip}(x,t,\beta(v_p - \frac{1}{p} + \eta(\frac{1}{p})) - \frac{1}{p}), \tag{4.8}$$

where

$$\beta(v_p - \frac{1}{p} + \eta(\frac{1}{p})) - \frac{1}{p} = (\beta_1(v_{1p} - \frac{1}{p} + \eta_1(\frac{1}{p})) - \frac{1}{p}), \cdots, (\beta_N(v_{N_p} - \frac{1}{p} + \eta_N(\frac{1}{p})) - \frac{1}{p}).$$

Multipling (4.8) by $(v_{ip} - \frac{1}{p})^r$ and integrating it over Ω,

$$\frac{d}{dt}\int_\Omega B(v_{ip} - \frac{1}{p} + \eta_i(\frac{1}{p}))dx + \frac{4r}{(1+r)^2}\int_\Omega |\nabla(v_{ip} - \frac{1}{p})^{\frac{r+1}{2}}|^2 dx$$
$$= \int_\Omega f_{ip}(x,t,\beta(v_p - \frac{1}{p} + \eta(\frac{1}{p}) - \frac{1}{p})(v_{ip} - \frac{1}{p})^r dx$$

here

$$B(s) = \int_{\eta_i(\frac{1}{p})}^s \eta_i'(\theta)(\theta - \eta_i(\frac{1}{p}))^r d\theta.$$

Condition (4.2) shows that the right side is less than

$$\int_\Omega |C_i(x,t)|(v_{ip}-\frac{1}{p})^r dx$$
$$+\sum_{j=1}^N \int_\Omega |C_{ij}(x,t)|\{\eta_j(\beta_j(v_{jp}-\frac{1}{p}+\eta_j(\frac{1}{p}))-\frac{1}{p}\}^{\alpha_{ij}}(v_{ip}-\frac{1}{p})^r dx$$
$$\le \int_\Omega |C_i(x,t)|(v_{ip}-\frac{1}{p})^r dx+\sum_{j=1}^N \int_\Omega |C_{ij}(x,t)|\{v_{jp}-\frac{1}{p}+\eta_j(\frac{1}{p})\}^{\alpha_{ij}}(v_{ip}-\frac{1}{p})^r dx$$
$$\le \int_\Omega [|C_i(x,t)|+\sum_{j=1}^N \int_\Omega |C_{ij}(x,t)|(\eta_j(\frac{1}{p}))^{\alpha_{ij}}](v_{ip}-\frac{1}{p})^r dx$$
$$+\int_\Omega \sum_{j=1}^N \int_\Omega |C_{ij}(x,t)|(v_{jp}-\frac{1}{p})^{\alpha_{ij}}(v_{ip}-\frac{1}{p})^r dx.$$

Therefore, applying Young's inequality we obtain the estimate

$$\frac{d}{dt}\int_\Omega B(v_{ip}-\frac{1}{p}+\eta_i(\frac{1}{p}))dx+\frac{4r}{(1+r)^2}\int_\Omega |\nabla(v_{ip}-\frac{1}{p})^{\frac{r+1}{2}}|^2 dx$$
$$\le \int_\Omega [|C_i(x,t)|+\sum_{j=1}^N |C_{ij}(x,t)|(\eta_j(\frac{1}{p}))^{\alpha_{ij}}]\frac{\alpha_{ij}}{1+\alpha_{ij}}dx$$
$$+\int_\Omega \{\sum_{j=1}^N |C_{ij}(x,t)|[\frac{\alpha_{ij}}{r+\alpha_{ij}}(v_{jp}-\frac{1}{p})^{\alpha_{ij}+r}+\frac{r_{ij}}{r+\alpha_{ij}}(v_{ip}-\frac{1}{p})^{\alpha_{ij}+r}]$$
$$+[|C_i(x,t)|+\sum_{j=1}^N |C_{ij}(x,t)|(\eta_j(\frac{1}{p}))^{\alpha_{ij}}\frac{r}{r+\alpha_{ij}}(v_{ip}-\frac{1}{p})^{\alpha_{ij}+r}]\}dx$$
$$\le C_i^*+\sum_{j=1}^N C_{ij}^*\int_\Omega (v_{jp}-\frac{1}{p})^{r+\alpha_{ij}}dx, \tag{4.9}$$

where C_i^*, C_{ij}^* are constants depending on $\|C_i\|_{L^\infty(Q_T)}, \|C_{ij}\|_{L^\infty(Q_T)}, \alpha_{ij}, r$.

Now set $r_{ij}=2\frac{r+\alpha_{ij}}{r+1}\in(1,\frac{2N}{N-2}), N\ge 3$ and apply the embedding theorem, we get

$$\int_\Omega (v_{ip}-\frac{1}{p})^{r+\alpha_{ij}}dx=\int_\Omega [(v_{jp}-\frac{1}{p})^{\frac{r+1}{2}}]^{r_{ij}}dx$$
$$\le C(n,\Omega)[\int_\Omega |\nabla(v_{jp}-\frac{1}{p})^{\frac{r+1}{2}}|^2 dx]^{\frac{r_{ij}}{2}}. \tag{4.10}$$

Substituting (4.10) into the right side of (4.9) we have

$$\frac{d}{dt}\int_\Omega B(v_{ip}-\frac{1}{p}+\eta_i(\frac{1}{p}))dx+\frac{4r}{(1+r)^2}\int_\Omega |\nabla(v_{ip}-\frac{1}{p})^{\frac{r+1}{2}}|^2 dx$$
$$\le C_i^*+\sum_{j=1}^N C(n,\Omega)C_{ij}^*[\int_\Omega |\nabla(v_{jp}-\frac{1}{p})^{\frac{r+1}{2}}|^2 dx]^{\frac{r_{ij}}{2}}.$$

Integrating with respect to t and using Holder's inequality and Young's inequality we obtain

$$\sum_{i=1}^{N}\int_0^T \frac{d}{dt}\int_\Omega B(v_{ip}-\frac{1}{p}+\eta_i(\frac{1}{p}))dxdt+\frac{4r}{(1+r)^2}\sum_{i=1}^{N}\int_{Q_T}|\nabla(v_{ip}-\frac{1}{p})^{\frac{r+1}{2}}|^2dxdt$$

$$\leq T\sum_{i=1}^{N}C_i^*+\sum_{i,j=1}^{N}C(n,\Omega)C_{ij}^*T^{1-\frac{r_{ij}}{2}}([\int_{Q_T}|\nabla(v_{jp}-\frac{1}{p})^{\frac{r+1}{2}}|^2dxdt)^{\frac{r_{ij}}{2}}.$$

$$\leq T\sum_{i=1}^{N}C_i^*+\sum_{i,j=1}^{N}C(n,\Omega)C_{ij}^*T^{1-\frac{r_{ij}}{2}}\{\frac{1}{\varepsilon^{r_{ij}/(2-r_{ij})}}(1-\frac{\alpha_{ij}}{2})$$

$$+\frac{r_{ij}}{2}\varepsilon\int_{Q_T}|\nabla(v_{jp}-\frac{1}{p})^{\frac{r+1}{2}}|^2dxdt\}.$$

Choose ε such that $\varepsilon\sum_{i,j=1}^{N}C(n,\Omega)C_{ij}^*T^{1-\frac{r_{ij}}{2}}\frac{r_{ij}}{2}\leq\frac{2r}{(1+r)^2}$, then

$$\sum_{i=1}^{N}\int_0^T\frac{d}{dt}B(v_{ip}-\frac{1}{p}+\eta_i(\frac{1}{p}))dxdt+\sum_{i=1}^{N}\frac{2r}{(1+r)^2}\int_{Q_T}|\nabla(v_{ip}-\frac{1}{p})^{\frac{r+1}{2}}|^2dxdt$$

$$\leq\tilde{C}(\alpha_{ij},r,N,n,|\Omega|,T,||C_i||_{L^\infty(Q_T)},||C_{ij}||_{L^\infty(Q_T)}).$$

Since $\beta'(s)\geq 0$ for $s\geq 0$, it follows:

$$\sum_{i=1}^{N}\int_{Q_T}|\nabla(v_{ip}-\frac{1}{p})^{\frac{r+1}{2}}|^2dxdt\leq C(\alpha_{ij},r,n,N,|\Omega|,T,||C_i||_{L^\infty(Q_T)},||C_{ij}||_{L^\infty(Q_T)}).$$

By Poincare's inequality we obtain the estimate

$$\sum_{i=1}^{N}||v_{ip}-\frac{1}{p}||_{L^{r+1}(Q_T)}\leq C. \tag{4.11}$$

In order to obtain the estimate of u_{ip}, multiply (4.2) by $(u_{ip}-k_i)^+(k_i\geq||\psi_i||_{L^\infty(\Omega)})$ and integrate it over $Q_\tau=\Omega\times(0,\tau)$ for $\tau\in(0,T]$,

$$\int_{Q_\tau}(u_{ip}-k_i)^+(u_{ip})_tdxdt+\int_{Q_\tau}\nabla(u_{ip}-k_i)^+\eta_i'(u_{ip})\nabla u_{ip}dxdt$$

$$=\int_{Q_\tau}f_{ip}(x,t,u_p-\frac{1}{p})(u_{ip}-k_i)^+dxdt. \tag{4,12}$$

Hypothesis (H_1) implies

$$\frac{1}{2}\int_\Omega[(u_{ip}-k)^+]^2dx+\varepsilon_i(k_i)\int_{Q_\tau}|\nabla(u_{ip}-k_i)^+|^2dx$$

$$\leq\int_{Q_\tau}C_i(x,t)(u_{ip}-k_i)^+dxdt+\int_{Q_\tau}\sum_{j=1}^{N}C_{ij}(x,t)[\eta_j(u_{jp}-\frac{1}{p})]^{\alpha_{ij}}(u_{ip}-k_i)^+dxdt$$

$$\leq \int_{Q_T} \{|C_i(x,t)| + \sum_{j=1}^{N} |C_{ij}(x,t)| [\eta_j(u_{jp})]^{\alpha_{ij}}\} (u_{ip} - k_i)^+ dxdt$$

$$\leq \int_{Q_T} \{|C_i(x,t)| + \sum_{j=1}^{N} |C_{ij}(x,t)| (\eta_j(\frac{1}{p}))^{\alpha_{ij}}$$

$$+ \sum_{j=1}^{N} |C_{ij}(x,t)| (v_{jp} - \frac{1}{p})^{\alpha_{ij}}\} (u_{ip} - k_i)^+ dxdt. \tag{4.13}$$

If we set

$$g_{ip}(x,t) = |C_i(x,t)| + \sum_{j=1}^{N} |C_{ij}(x,t)| (\eta_j(\frac{1}{p}))^{\alpha_{ij}} + \sum_{j=1}^{N} |C_{ij}(x,t)| (v_{jp} - \frac{1}{p})^{\alpha_{ij}}.$$

Then by the above argument we have

$$\|g_{ip}\|_{L^q(Q_T)} \leq \hat{C}(q, \text{ data})$$

for any $q < \infty$. Fix some $q \in (\frac{N+2}{2}, \infty)$ and observe the norm in space $\overset{\circ}{V}_2\ (Q_T)$ [10]. It follows

$$\begin{aligned} \|(u_{ip} - k_i)^+\|^2_{\overset{\circ}{V}_2(Q_T)} &\leq C \int_{Q_T} |g_{ip}| (u_{ip} - k_i)^+ dxdt \leq C \|g_{ip}\|_{L^q(Q_T)} \|(u_{ip} - k_i)^+ \chi\|_{L^{q'}(Q_T)}, \\ &\leq C \|g_{ip}\|_{L^q(Q_T)} \|(u_{ip} - k_i)^+\|_{L^{\frac{2(N+2)}{N}}(Q_T)} \|\chi\|_{L^r(Q_T)} \end{aligned}$$

where χ be the characteristic function for set $\{(x,t) | u_{ip} > k_i, (x,t) \in Q_T\}$, $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{r} = \frac{1}{2}(1 + \frac{2}{N+2} - \frac{2}{q})$.

Using embedding theorem (see [10] Chap.II)

$$V_2(Q_T) \subset L^{\frac{2(N+2)}{N}}(Q_T),$$

we get

$$\|(u_{ip} - k_i)^+\|_{V_2(Q_T)} \leq \nu M(p, k_i)^{1/r},$$

where ν is a constant independent on p and $M(p, k_i)$ denotes the measure of set $\{(x,t) | u_{ip} > k_i, (x,t) \in Q_T\}$.

It follows directly from Theorem 6.1 in [10] Chap.II, there exist a constant C such that

$$u_{ip}(x,t) \leq C \text{ in } Q, \tag{4.14}$$

for all p. This completes the proof.

The remainder of the proof is the same as in [5], that completes the proof of theorem 2.

Remark Under some assumptions for unbounded domain $\Omega \subset \mathbf{R}^n$ the conclusion of Theorem 2 hold.

We investigate global nonexistence of solution. We show out if there is f_i such that its growth is too rapid than some η_i, then the solution of problem (2.1)–(2.3) will be blow-up in finite time. We need the following hypothesis:

(H_2') $f_i(u_1, u_2, \ldots, u_N) \equiv f_i(u_i, V_i, W_i) \in C'(\mathbf{R}_+^N)(i = 1, 2, \ldots, N)$ is quasimonotone nondecreasing in component u_i and V_i, and nonincreasing in component W_i. There is at least one of $k \in \{1, 2, \ldots, N\}$ such that

$$\theta_k = f_k(u_1, \cdots, u_{k-1}, u_k, u_{k+1}, \cdots, u_N)/\eta_k(u_k) \tag{4.15}$$

is nondecreasing in u_k on $\mathbf{R}^+ = [0, \infty)$.

(H_4) There exist $\kappa \in (0, \frac{1}{2})$ such that Φ^κ is a convex frnction on $\mathbf{R}^+$, here

$$\Phi_k(s) = \int_0^s \eta_k(\theta) d\theta. \tag{4.16}$$

Theorem 3 *Let $(H_1), (H_2'), (H_3), (H_4)$ lold, $\eta_k(\psi_k) \in H_0^1(\Omega)$ and*

$$\int_\Omega P_k(\psi_k) dx > \frac{1}{2} \int_\Omega |\nabla \eta_k(\psi_k)|^2 dx - \frac{3 - 4\kappa}{(1 - 2\kappa)^2} \frac{1}{T} \int_\Omega \Phi_k(\psi_k) dx. \tag{4.17}$$

Then problem (2.1)-(2.3) with $f_i(u)$ has not any bounded solutions. Moreover, if there exist $t_1 \in (0, T]$ and $u = (u_1, u_2, \ldots, u_N)$ is a solution in $Q_{t_1} = \Omega \times (0, t_1)$, then

$$\lim_{t \to t_1^-} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty, \tag{4.18}$$

where

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} = \sum_{i=1}^N u_i(\cdot, t)\|_{L^\infty(\Omega)}$$

and

$$P_k(u_k) = \int_0^{u_k} \eta_k(s) f_k(u_1, \cdots, u_{k-1}, s, u_{k+1}, \cdots, u_N) ds. \tag{4.19}$$

The proof of this theorem is same as in the proof of corresponding theorem in [5], [8].

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非线性退缩反应扩散方程组的整体解

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摘 要

本文讨论非线性退缩方程组

$$u_{it} = \Delta\eta_i(u_i) + f_i(x,t,u_1,\cdots,u_N),(x,t) \in Q_T = \Omega \times (0,T)$$

具有 Dirichlet 边界条件的解之整体存在和非整体存在.