Journal of Mathematical Research & Exposition
Vol.13, No.1, Feb. 1993

Co-RS-compact Topologies*

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Abstract. A topoogy $R(\tau)$ is constructed from a given topology τ on a set X. $R(\tau)$ is coarser than τ, and the following are some results based on this topology:
1. Continuity and RS-continuity are equivalent if the codomain is retopologized by $R(\tau)$.
2. Each topological space has a coarser extremely disconnected topology.
3. The class of semi-open sets with respect to $R(\tau)$ is a topology.
4. T_2 and semi-T_2 properties are equivalent on a space whose topology is $R(\tau)$.
5. Minimal R_0−spaces are RS-compact.
6. Maximal α−compact spaces are RS-compact.

1. Introduction

Throughout the present paper (X,σ) and (Y,τ) are topological spaces on which no separation axioms are assumed unless explicitly stated. A set S is said to be regular open (resp. regular closed) if $S = \text{int}(\text{cl}(S))$ (resp.$S = \text{cl}(\text{ int}(S))$. A set S is said to be α-open [17], if $S \subset \text{int}(\text{cl int}(S))$. A set S is said to be regular semi-open [4] (resp. semi-open [12]) if there exists a regular open (resp. open) set O such that $O \subset S \subset \text{cl} -(O)$. It should be noticed that the complement of a regular semi-open set is also regular semi-open. The family of all regular semi-opdn (resp. regular open, regular closed, α-open and semi-open) sets in X is denoted by RSO(X) (resp. RO(X), RC(X), $\alpha O(x)$, SO(X)). A space X is said to be extremely disconnected if for every open set O of X, cl (O) is open in X.

In 1980, Hong [8] has introduced a new class of topological spaces called RS- compact spaces which are characterized by the following property"Every regular closed cover has a finite subfamily, the interiors of whose members cover X".

Note The definition of RS-compact space in the sense of Hong is equivalent to that of an I-compact space in the sense of Cameron [5]. In 1985 Noiri. [18] has introduced RS-compact relative to X."A subset S of X is RS-compact relative to X if for every cover $\{V_i : i \in I\}$ of S by regular closed sets of X, there exists a finite subset I_0 of I such that $S \subset \cup\{\text{int}(V_i) : i \in I_0\}$.

In 1989, Abd El-Mondef et al., [2] have introduced RS-continuous function"A function $f : X \to Y$ is called RS-continuous if for each $x \in X$ and each open set $V \subset Y$ containing

*Received Aug. 19, 1991.

$f(x)$ having RS-compact complement, there exists an open set $U \subset X$ containing x such that $f(U) \subset V$."A space X is said to be almost normal space [14] if for every two disjoint regular closed subsets F_1 and F_2 of X, there exist two disjoint open sets U and V in X such that $F_1 \subset U$ and $F_2 \subset V$. A space X is semi-T_2 [13] (resp. semi-T_2' [1]) if for each $x, y \in X, x \neq y$, there exist U and $V \in$SO(X) such that $x \in U, y \in V$, and $U \cap V = \emptyset$ (resp. cl$(U) \cap$cl$(V) = \emptyset$). The space (Y, τ) is R_0 [6] if for each $G \in \tau, x \in G$ implies cl$\{x\} \subseteq G$. We observe that in every R_0 topological space the closure of a singleton set is compact.

A space X is α-compact [3] if each cover of X by α-open sets in X has a finite subcover.

Theorem 1.1[18] *If $A \in RSO(X)$ and B is RS-compact relative to X, then $A \cup B$ is RS-compact relative to X.*

Theorem 1.2[18] *Let $A \in RO(X)$. Then A is RS-compact relative to X.*

Theorem 1.3[18] *If X is RS-compact and $A \in RO(X)$, then A is RS-compact.*

Theorem 1.4[18] *Let X_0 be an open set of X. Then we have:*

(1) If $A \subset X_0$, then $Int_X(A) = Int_{X_0}(A)$.

(2) If $V \in RSO(X)$, then $V \cap X_0 \in RSO(X_0)$.

Theorem 1.5[5] *Any RS-compact is extremely disconnected.*

Theoren. 1.6[9] *Every compact Hausdorff space is normal.*

Theorem 1.7[11] *A space (X, τ) is semi-T_2' iff for each $x, y \in X, x \neq y$, there exist $W_{1'}, W_2 \in RC(X, \tau)$ containing x, y respectively, such that $W_1 \cap W_2 = \emptyset$.*

Theorem 1.8[10] *The following statements are equivalent for a space (X, τ):*

(a) (X, τ) is extremely disconnected.

(b) For each $A \in SO(X, \tau), cl(A) \in \tau$.

(c) For each $A, B \in SO(X, \tau), cl(A \cap B) = cl(A) \cap cl(B)$.

2. Co RS-compact topologies

Let (Y, τ) be a topological space, and consider $R'(\tau) = \{U \in \tau : Y - U$ is RS-compact relative to $\tau\}$. $R'(\tau)$ is a base for a topology $R(\tau)$ on Y, called the Co RS-compact topology on Y. We shall denote by $(Y, R(\tau))$ to be a Co RS-compact space of (Y, τ), and R-cl (S) (resp. R-int (S)) will denote the closure (resp. interior) with respect to $R(\tau)$ of a subset S of $(Y, R(\tau))$. From definition we have $R(\tau) \subset \tau$, and the following lemma is a direct consequance.

Lemma 2.1 *The function $f : (X, \sigma) \to (Y, \tau)$ is RS-continuous iff $f : (X, \sigma) \to (Y, R(\tau))$ is continuous.*

Theorem 2.1 *For any topological space $(Y, \tau), (Y, R(\tau))$ is RS-compact space.*

Proof Consider the $R(\tau)$ regular closed cover $\Delta = \{V_i : i \in I\}$ of Y, and V be some nonemptay member of Δ, there exists $R(\tau)$ open set U such that $U \subset V \subset R$-cl (U), and

$Y-U$ is RS-compact relative to τ. By using Theorem 1.1 and Theorem 1.2 $(Y-V)$, is RS-compact subspace of Y. Assume that $A=Y-V$, thus $V_i\cap A$ $\in$RSO(A), for each $i\in I$ (By using Theorem 1.4,and $A=\cup\{V_i\cap A : i\in I\}$. Thete exists a finite subset I_0 of I such that $A=\cup\{\text{int}_A(V_i\cap A) : i\in I\}$. From Theorem 1.4, we have $\text{int}_A(V_i\cap A)=\text{int}_Y(V_i\cap A)\subset\text{int}_Y(V_i)$, for each $i\in I_0$. Hence $A\subset\cup\{\text{int}_Y(V_i) : i\in I_0\}$, and $A=Y-V$ is RS-compact relative to $R(\tau)$. Thus $(Y,R(\tau))$ is RS-compact space.

Proposition 2.1 *The following are hold:*

(a) $RO(Y,R(\tau))\subset RO(Y,\tau)$.

(b) $cl_\tau G=cl_R G$, *for all* $G\in RO(Y,R(\tau))$.

(c) $RSO(Y,R(\tau))\subset RO(Y,\tau)$.

Proof (a) Let $G\in$RO$(Y,R(\tau))$, then $G=\text{int}_R\text{cl}_R(G)=\text{cl}_R(G)[(Y,R(\tau))$ is extremely disconnected]. But $\text{cl}_\tau G\subset\text{cl}_R G$, implies that $G=\text{cl}_\tau G$ and $G=\text{int}_\tau\text{cl}_\tau G$.

(b) From (a), the proof is obvious.

(c) By using (b), the result follows.

Proposition 2.2 *Let X be an extremely disconnected space then:*

(a) The union of a finite regular open sets is a regular open set.

(b) If X is the union of a finite number of regular open RS-compact subspaces, then X is RS-compact space.

Lemma 2.2 *If* $(Y,R(\tau))$ *is almost normal space, then* (Y,τ) *is RS-compact.*

Proof Let F_1 and F_2 be two disjoint $R(\tau)$ regular closed sets, then there exist disjoint $R(\tau)$ open sets U and V such that $F_1\subset U$ and $F_2\subset V$. Hence $Y=Y-(U\cap V)=(Y-U)\cup(Y-V)=(Y-F_1)\cup(Y-F_2)$. By using Proposition 2.1 and Theorem 2.1 and Proposition 2.2, we arrive (Y,τ) is RS-compact.

Theorem 2.2 (Y,τ) *is RS-compact iff* $\tau=R(\tau)$.

Proof Let $\tau=R(\tau)$, then (Y,τ) is RS-compact. Conversely, we assume that (Y,τ) is RS-compact, to prove that $\tau\subset R(\tau)$. Let $U\in\tau$, then $\text{cl}_\tau U\in\tau$, and $Y-\text{cl}_\tau U\in$RO(Y,τ), which implies that $Y-\text{cl}_\tau U$ is RS-compact relative to τ. Hence $\text{cl}_\tau U\in R(\tau)$ and $U\in R(\tau)$.

3. Separation Properties

The property of being T_1 space is expansive, that is, if (Y,τ) is T_1 and $\tau\subset\tau'$ then (Y,τ') is T_1 but it is not generally contractive. The following result proves the contractivity of T_1 property from (Y,τ) to $(Y,R(\tau))$.

Lemma 3.1 *If* (Y,τ) *is* T_1, *then* $(Y,r(\tau))$ *is* T_1.

Proof Let x be any point of Y, then $\{x\}$ is closed in τ and RS-compact in (Y,τ). Thus $Y-\{x\}$ is open in τ and $\{x\}$ is RS-compact. Hence $Y-\{x\}$ is open in $R(\tau)$. Thus $(Y,R(\tau))$ is T_1.

Theorem 3.1 *If* (Y,τ) *is Hausdorff, then* $(Y,R(\tau))$ *is compact.*

Proof From Lemma (2) in [16], we have that $R(\tau) \subset c(\tau_s) \subset \tau_s \subset n(\tau) \subset \tau$. Using Lemma (4) in [15] and Theorem 2.1 in [19], the result follows.

Lemma 3.2 *If $(Y, R(\tau))$ is Hausdorff, then (Y, τ) is normal.*

Proof By using Theorem (3.1), the proof is obvious.

Lemma 3.3 *If $(Y, R(\tau))$ is semi-T_2', then (Y, τ) is RS-compact space.*

Corollary 3.1 *If $(Y, R(\tau))$ is semi-T_2', then $RO(Y, R(\tau)) = RO(Y, \tau)$.*

Lemma 3.4 *If $(Y, R(\tau))$ is semi-T_2', then $\alpha O(Y, R(\tau)) = \alpha O(Y, \tau)$.*

Proof Let $G \in \alpha O(R(\tau))$, then $G \subset \text{int}_R \text{cl}_R \text{int}_R G \subset \text{int}_\tau \text{cl}_R \text{int}_\tau G = \text{int}_\tau \text{cl}_\tau \text{int}_\tau G$. Hence $G \in \alpha O(\tau)$. Conversely. If $G \in \alpha O(\tau)$, then $G \subset \text{int}_\tau \text{cl}_\tau \text{int}_\tau G = \text{int}_R \text{cl}_R \text{int}_R G$, then $G \in \alpha O(R(\tau))$.

Corollary 3.2 *If $(Y, R(\tau))$ is semi-T_2', then $(Y, R(\tau))$ is α-compact iff (Y, τ) is α-compact.*

Corollary 3.3 *Maximal α-compact spaces are RS-compact.*

Theorem 3.2 *$(Y, R(\tau))$ is semi-T_2 iff $(Y, R(\tau))$ is semi-T_2'.*

Proof Let $(Y, R(\tau))$ be semi-T_2, and $x, y \in Y, x \neq y$, then there exist $U, V \in \text{SO}(Y, \tau)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$, which implies that $\text{cl}_R(U \cap V) = \emptyset$. Since $(Y, R(\tau))$ is extremely disconnected we have $\text{cl}_R \text{int}_R U = \text{cl}_R U \in \tau$, and $\text{cl}_R \text{int}_R V = \text{cl}_R V \in \tau$. But $\text{cl}_R U \cap \text{cl}_R V = \text{cl}_R(U \cap V) = \emptyset$. Hence $(Y, R(\tau))$ is semi-T_2'. The converse is obvious.

Theorem 3.3 *$(Y, R(\tau))$ is Haussdorff iff $(Y, R(\tau))$ is semi-T_2.*

Proof It is simillar to the proof of Theorem 3.2.

Theorem 3.4 *Let (Y, τ) be a space, then:*

(a) The class of $SO(Y, R(\tau))$ form a topology (denoted by τ') finer than $R(\tau)$.

(b) $RO(Y, R(\tau)) = RO(Y, \tau')$.

(c) (Y, τ') is RS-compact.

Proof (a) Since $(Y, R(\tau))$ is RS-compact, it is extremely disconnected and hence SO(Y, $R(\tau)$) forms a topology such that $R(\tau) \subset \tau'$.

(b) Let $G \in \tau'$. Then $\text{cl}_{\tau'} G \subset \text{cl}_R G$. Conversely, let $x \in \text{cl}_R G$, and $x \in U, U \in \tau'$. Hence $x \in \text{cl}_R\ \text{int}_R U \in R(\tau)$ and $\text{cl}_R\ \text{int}_R U \cap \neq \emptyset$. But $G \in \tau'$, which implies $G \in \text{SO}(Y, R(\tau))$ and $G \subset \text{cl}_R \text{int}_R G$. Therefore $\emptyset \neq \text{cl}_R \cap \text{cl}_R \text{int}_R U \subset (\text{int}_R G \cap \text{cl}_R \text{int}_R U) \subset \text{cl}_R(\text{int}_R G \cap \text{int}_R U) \subset \text{cl}_R(U \cap G)$, which implies that $U \cap G \neq \emptyset$, and so $x \in \text{cl}_{\tau'} G$. Hence $\text{cl}_R G \subset \text{cl}_{\tau'} G$. Thus $\text{cl}_R = \text{cl}_{\tau'} G$ for each $G \in \tau'$.

(c) Using (b), the result follows.

Lemma 3.5 *If (Y, τ) is R_0, then $(Y, R(\tau))$ is R_0.*

Proof Let $x \in Y$, and $x \in G \in R(\tau)$. Then $G \in \tau$, and $\text{cl}_\tau\{x\} \subset G$. Since $\text{cl}_\tau\{x\}$ is compact in (Y, τ), implies that it is compact in $(Y, R(\tau))$, and nearly compact in $(Y, R(\tau))$. But $(Y, R(\tau))$ is extremely disconnected, then $\text{cl}_\tau\{x\}$ is RS-compact relative to $R(\tau)$, which implies that it is RS-compact relative to τ. Thus $\mathrm{cl}\{x\}$ is closed in $(Y, R(\tau))$, implies that $\mathrm{cl}_R\{x\} \subset \mathrm{cl}_\tau\{x\}$. Hence $\mathrm{cl}_R\{x\} = \mathrm{cl}_\tau\{x\}$, and $(Y, R(\tau))$ is R_0.

Theorem 3.5 *Minimal RO spaces are RS−compact spaces.*

Proof Using Lemma 3.5, the result follows.

References

[1] M.E. Abd El-Monsef,*Studies on some pretopological concepts*, Ph.D. Thesis, Tanta Univ., 1980.

[2] M.E. Abd El-Monsef, R.A. Mahmoud, A.A. Nasef, *Functions based on compactness*, to appe -ar.

[3] R.H. Atia, S.N. El-Deeb, A.S. Mashhour,*$\alpha-$compactness and $\alpha-$homeomorphism*, preprint.

[4] D.E. Cameron, *Properties of $S-$closed spaces*, Proc. Amer. Math. Soc., **72(3)**(1978), 581–585.

[5] D.E. Cameron,*Some maximal topologies which are QHC*, Proc. Amer. Math. Soc., **75(1)**(1979), 149–156.

[6] A.S. Davis,*Indexed systems of neighborhoods for general topological spaces*, Amer. Math. Monthly, **68**(1961), 886–893.

[7] D.B. Gauld, M. Mrsevic, I.L. Reilly, M.K. Vamanamurthy, *Colindelöf topologies and L-continuous functions*, GLASNIK Math., **19**(39)(1984), 297–308.

[8] W.C. Hong, *RS-compact spaces*, J. Korean Math. Soc., **17**(1980), 39–43.

[9] S.T. Hu, *Elements of general topology*, Holden Day, IM, 1972.

[10] D.S. Jankovic, *On locally irreducible spaces*, Ann. de la Soc. Sci. de Bruxelles, T, **97**, II, pp. 59-72, 1983.

[11] A.M. Kozae, *Studies on some maximal and minimal topological consepts*, Ph.D. Thesis, Tanta Univ., 1988.

[12] N. Lèvine, *Semi-open sets and semi continuity in topological spaces*, Amer. Math. Monthly, **70**(1963), 36–41.

[13] S.N. Maheshwari, R. Prasad, *Some new separation axioms*, Ann. Soc. Sci. Bruxelles, T. **3(89)**(1975), 395–407.

[14] A.S. Mashhour, F.S. Mahmoud, I.A. Hasanein, M.A. Fath Alla, *On some generalizations of compactness*, to appear.

[15] M. Mrsevic, I.L. Reilly, M.K. Vamanamurthy, *On semi- regularization topologies*, J. Autstral Math. Soc., (Series. A), **38**(1985), 40–54.

[16] M. Mrsevic, I.L. Reilly, *On $N-$continuity and $Co-N-$closed topologies*, Ricerche di Math., Vol. XXXVI, fasc I^0, 1987.

[17] O. Njasted, *On some classes of nearly open sets*, Pac. J. of Math., **15(3)**(1965), 961–970.

[18] T. Noiri, *On $RS-$compact spaces*, J. Korean Math. Soc., **22**(1985), No. 1, pp.19–34.

[19] M.K. Singal, A. Mathur, *On nearly compact spaces II*, Boll. Della Un. Math. Italiana, **9(4)**(1974), 670–678.