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On α-Close-to-Convex Functions of Order β *

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Abstract This paper is devoted to the investigation of the class $C(\alpha,\beta)$ which proves to be a subclass of C, the class of functions close-to-convex in unit disc.

1. Introduction

Let D be the unit disc in the complex plane, A the class of functions analytic in D and S the class of normal univalent analytic functions. Denote

$$M_\alpha=\{f: f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in A, \frac{1}{z}f(z)f'(z)\neq 0$$

and

$$\mathrm{Re}[(1-\alpha)\frac{zf'(z)}{f'(z)}+\alpha\frac{(zf'(z))'}{f'(z)}]>0\}$$

where α is a real number. Miller, Mocanu and Reade [3], [5] studied this class. They proved that all functions in M_α are starlike with respect to the origine. Let S^* be the starlike subclass of S, then $M_\alpha\subset S^*$.

Now we define a larger class

$$C(\alpha,\beta)=\{f(z)=z+\sum_{n=2}^{\infty}a_nz^n \text{ there exists some } g\in S^* \text{ and real } \theta\in(-\frac{\pi}{2},\frac{\pi}{2})$$

$$\text{such that } \mathrm{Re}[(1-\alpha)e^{i\theta}\frac{zf'(z)}{g(z)}+\alpha e^{i\theta}\frac{(zf'(z))'}{g'(z)}]>\beta\cos\theta(z\in D)\}$$

where $0\le\alpha\le 1, 0\le\beta<1$. In this paper we will prove that if $f(z)\in C(\alpha,\beta)$, then $f(z)$ must be close-to-convex. Obviously, $C(0,0)$ coincides with C, the class of close-to-convex functions. We also give a integral representation for this class. Finally, we obtain sharp estimates for the first three coefficients.

2. Results and their proofs.

Lemma 1[1] *Let $w(z)$ be analytic in $D, w^{(k)}(0)=0(0\le k\le p-1)$. Suppose there exists*

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a point $z_0 \in D$ *such that* $|w(z_0)| = \max_{|z|\le|z_0|} |w(z)|$, *then* $\{z_0 w'(z_0)/w(z_0)\} \ge p \ge 1$.

Lemma 2 *Let* $M(z) \in A$ *satisfying* $M(0) = M'(0) - 1 = 0, N(z) \in S^*$ *and* $0 \le \alpha \le 1, 0 \le \beta < 1$. If

$$(1) \qquad \mathrm{Re}\{e^{i\theta_0}[(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)}]\} > \beta\cos\theta_0, \quad (z \in D, \theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}), \theta_0 \text{ fixed }),$$

then we have

$$(2) \qquad \mathrm{Re}\{e^{i\theta_0}\frac{M(z)}{N(z)}\} > \beta\cos\theta_0.$$

Proof Set

$$\begin{aligned} e^{i\theta_0}\frac{M(z)}{N(z)} &= \frac{1+(1-2\beta)w(z)}{1-w(z)}\cos\theta_0 + i\sin\theta_0, \\ \varphi(z) &= e^{i\theta_0}\{(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)} - \beta\},\ w(0) = 0. \end{aligned}$$

Then we only have to show $|w(z)| < 1$. If there are some $z_0 \in D$ such that $|w(z_0)| = 1$, by Lemma 1, we have $z_0 w'(z_0) = k w(z_0)$, where $k \ge 1$. But

$$e^{i\theta_0}\frac{M'(z)}{N'(z)} = 2(1-\beta)\cos\theta_0\frac{w'(z)}{(1-w(z))^2}\frac{N(z)}{N'(z)} + \cos\theta_0\frac{1+(1-2\beta)w(z)}{1-w(z)} + i\sin\theta_0,$$

$$\varphi(z) = \cos\theta_0\frac{1+(1-2\beta)w(z)}{1-w(z)} + 2\alpha(1-\beta)\cos\theta_0\frac{w'(z)}{(1-w(z))^2}\frac{N(z)}{N'(z)} + i\sin\theta_0 - \beta e^{i\theta_0}.$$

Notice that

$$\frac{z_0 w'(z_0)}{(1-w(z_0))^2} = \frac{k w(z_0)}{1-w(z_0)} = \frac{k w(z_0)\overline{(1-w(z_0))^2}}{|1-w(z_0)|^4} = \frac{2k(\mathrm{Re}w(z_0)-1)}{|1-w(z_0)|^4} \le 0.$$

Hence we have

$$\mathrm{Re}\varphi(z_0) = (1-\beta)\cos\theta_0\mathrm{Re}\frac{1+w(z_0)}{1-w(z_0)} + 2\alpha(1-\beta)\cos\theta_0\frac{z_0 w'(z_0)}{(1-w(z_0))^2}\mathrm{Re}\frac{N(z_0)}{z_0 N'(z_0)} \le 0.$$

This provides the contradiction of condition (1).

Some special cases of this lemma were proved by K.Sakaguchi [6], Milier and Mocanu [4].

Theorem 1 *Let* $0 \le \alpha_2 \le \alpha_1 \le 1, 0 \le \beta_2 \le \beta < 1$, *then* $C(\alpha_1,\beta_1) \subset C(\alpha_2,\beta_2)$. *In particular,* $C(\alpha,\beta) \subset C(0,0)$, *i.e., every* $f(z) \in C(\alpha,\beta)$ *is close-to-convex.*

proof It is clear that $C(\alpha_1,\beta_1) \subset C(\alpha_1,\beta_2)$. Thus we need only to prove $C(\alpha_1,\beta_2) \subset C(\alpha_2,\beta_2)$.

Case i $\alpha_2 = 0$. By the definition of $C(\alpha_1,\beta_2)$ and Lemma 2, we obtain our result immediately.

Case ii $\alpha_2 \neq 0$. Suppose $f(z) \in C(\alpha_1, \beta_2)$, by means of Lemma 2, we have

$$\begin{aligned}
&\mathrm{Re}\{(1-\alpha_2)e^{i\theta}\frac{zf'(z)}{g(z)} + \alpha_2 e^{i\theta}\frac{(zf'(z))'}{g'(z)}\} \\
&= \frac{\alpha_2}{\alpha_1}\mathrm{Re}\{(\frac{\alpha_1}{\alpha_2}-1)e^{i\theta}\frac{zf'(z)}{g(z)} + [(1-\alpha_1)e^{i\theta}\frac{zf'(z)}{g(z)} + \alpha_1 e^{i\theta}\frac{(zf'(z))'}{g'(z)}]\} \\
&> \frac{\alpha_2}{\alpha_1}\{(\frac{\alpha_2}{\alpha_2}-1)\beta_2\cos\theta + \beta_2\cos\theta\} = \beta_2\cos\theta.
\end{aligned}$$

Thus we have $f(z) \in C(\alpha_2, \beta_2)$.

Theorem 2 *$f(z) \in C(\alpha, \beta)$ if and only if there are $\theta, g(z) \in S^*$ and Schwarz function $w(z)$($w(z)$ analytic in $D, w(0)=0$ and $|w(z)|<1$) such that*

$$(3) \qquad f'(z) = \frac{1}{\alpha z g'(z)^{\frac{1}{\alpha}-1}}\int_0^z \frac{1+[(1-\beta)e^{-2i\theta}-\beta]w(z)}{1-w(z)} g(z)^{\frac{1}{\alpha}-1}g'(z)dz, \text{ if } 0<\alpha\leq 1,$$

or

$$(4) \qquad f'(z) = \frac{g(z)}{z}\frac{1+[(1-\beta)e^{-2i\theta}-\beta]w(z)}{1-w(z)}, \text{ if } \alpha = 0.$$

Proof If $f(z) \in C(\alpha,\beta)(\alpha \neq 0)$, then there is a Schwarz function $w(z)$ satisfies

$$[(1-\alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)}]e^{i\theta} = \frac{1+(1-2\beta)w(z)}{1-w(z)}\cos\theta + i\sin\theta,$$

where $g(z)$ and θ are the same as described in the definition. Hence

$$(5) \qquad (1-\alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)} = \frac{1+[(1-\beta)e^{-2i\theta}-\beta]w(z)}{1-w(z)}$$

Multiply $\frac{1}{\alpha}g(z)^{\frac{1}{\alpha}-1}g'(z)$ on both sides, we have

$$\begin{aligned}
&(\frac{1}{\alpha}-1)zf'(z)g(z)^{\frac{1}{\alpha}-2}g'(z) + (zf'(z))'g(z)^{\frac{1}{\alpha}-1} \\
&= \frac{1+[(1-\beta)e^{-2i\theta}-\beta]w(z)}{1-w(z)}\frac{1}{\alpha}g(z)^{\frac{1}{\alpha}-1}g'(z).
\end{aligned}$$

Note that the left of above formula is exactly $\frac{d}{dz}[zf'(z)g(z)^{\frac{1}{\alpha}-1}]$. Thus we obtain

$$zf'(z)g(z)^{\frac{1}{\alpha}-1} = \frac{1}{\alpha}\int_0^z \frac{1+[(1-\beta)e^{-2i\theta}-\beta]w(z)}{1-w(z)}g(z)^{\frac{1}{\alpha}-1}g'(z)dz,$$

which is equivalent to (3).

If $f(z) \in C(0,\beta)$, then (4) is the immediatelly consequence of (5). Conversely, if $f(z)$ satisfies (3) or (4) we can easily prove $f(z) \in C(\alpha,\beta)$.

Let $\theta = 0, w(z) = z$ and $g(z) = \frac{z}{(1-z)^2}$, the corresponding function $f_0(z)$ will be the extremal function in the following theorem.

Theorem 3 *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in C(\alpha,\beta)$, then we have sharp estimates*

(6) $$|a_2|\le\frac{2+\alpha-\beta}{1+\alpha},$$

(7) $$|a_3|\le\frac{6\alpha^2+23\alpha+9-2\beta(7\alpha+3)}{3(1+\alpha)(1+2\alpha)},$$

(8) $$|a_4|\le\frac{(1+\alpha)(1+2\alpha)(1+3\alpha)+(1-\beta)(23\alpha^2+16\alpha+3)}{(1+\alpha)(1+2\alpha)(1+3\alpha)}.$$

$f_0(z)$ is the extremal function.

Lemma 3 *Let $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in S^*$ and $\mu\le\frac{1}{2}$, then*

(9) $$|b_3-\mu b_2^2|\le 3-4\mu.$$

Koebe function $K(z)=\frac{z}{(1-z)^2}$ is the extremal function (see [2]).

Proof Let $p(z)=\dfrac{1+w(z)}{1-w(z)}=1+\sum_{n=1}^{\infty}p_nz^n, g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in S^*$ in (5), we have

$$\begin{aligned}&(1-\alpha)(z+2a_2z^2+3a_3z^3+4a_4z^4+\cdots)(1+2b_2z+3b_3z^2+4b_4z^3+\cdots)\\&\quad+\alpha(1+4a_2z+9a_3z^2+16a_4z^3+\cdots)(z+b_2z^2+b_3z^3+b_4z^4+\cdots)\\&=(1+p_1'z+p_2'z^2+p_3'z^3+\cdots)(z+b_2z^2+b_3z^3\\&\quad+b_4z^4+\cdots)(1+2b_2z+3b_3z^2+4b_4z^3+\cdots),\end{aligned}$$

where $p_n'=(1-\beta)e^{-i\theta}\cos\theta p_n$. Compare the coefficients of z^2, z^3 and z^4, we obtain

(10) $$2(1+\alpha)a_2=(1+\alpha)b_2+p_1',$$

(11) $$3(1+2\alpha)a_3=-4a_2b_2+(1+2\alpha)b_3+2b_2^2+3b_2p_1'+p_2',$$

(12) $$\begin{aligned}4(1+3\alpha)a_4=&-(6+3\alpha)a_3b_2-(6-2\alpha)b_3a_2+5b_2b_3\\&+(1+3\alpha)b_4+(4b_3+2b_2^2)p_1'+3b_2p_2'+p_3'.\end{aligned}$$

Since $|p_n'|\le 2(1-\beta), |b_n|\le n$, we obtain (6) from (10) at once. Substitute a_2 in (11), we obtain (7)

$$|a_3|\le\frac{6\alpha^2+23\alpha+9-2\beta(7\alpha+3)}{3(1+\alpha)(1+2\alpha)}.$$

Substitute a_2, a_3 in (12),we have

$$\begin{aligned}4(1+\alpha)(1+2\alpha)(1+3\alpha)a_4 &= (1+\alpha)(1+2\alpha)(1+3\alpha)b_4+\alpha(\alpha-1)b_2^2p_1'\\ &\quad+(1+2\alpha)(1+5\alpha)b_3p_1'+(1+\alpha)(1+5\alpha)b_2p_2'+(1+\alpha)(1+2\alpha)p_3'\\ &=(1+\alpha)(1+2\alpha)(1+3\alpha)b_4+(1+2\alpha)(1+5\alpha)p_1'[b_3-\frac{\alpha(1-\alpha)}{(1+2\alpha)(1+5\alpha)}b_2^2]\\ &\quad+(1+\alpha)(1+5\alpha)b_2p_2'+(1+\alpha)(1+2\alpha)p_3'\end{aligned}$$

Let $\mu=\frac{\alpha(1-\alpha)}{(1+2\alpha)(1+5\alpha)}$, then $\mu\le\frac{1}{2}$ when $\alpha\ge 0$. By means of Lemma 3, we have

$$|b_3-\frac{\alpha(1-\alpha)}{(1+2\alpha)(1+5\alpha)}b_2^2|\le\frac{34\alpha^2+17\alpha+3}{(1+2\alpha)(1+5\alpha)}.$$

Hence

$$\begin{aligned}4(1+\alpha)(1+2\alpha)(1+3\alpha)|a_4| &\le 4(1+\alpha)(1+2\alpha)(1+3\alpha)\\ &\quad+2(1-\beta)(34\alpha^2+17\alpha+3)+4(1-\beta)(1+5\alpha)(1+\alpha)+\\ &\quad 2(1-\beta)(1+\alpha)(1+2\alpha)\\ |a_4| &\le\frac{(1+\alpha)(1+2\alpha)(1+3\alpha)+(1-\beta)(23\alpha^2+16\alpha+3)}{(1+\alpha)(1+2\alpha)(1+3\alpha)}.\end{aligned}$$

From the proof we can see $f_0(z)$ is the extremal function.

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关于β级α近于凸函数

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摘　要

本文研究了单位圆盘上的近于凸函数族子族的若干问题.