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On 3-Dimensional Cooperative Systems in the Box [p,q]*

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1. Introduction

Consider a C^1 system of differential equations defined on N

$$\dot{x} = F(x), \tag{1}$$

where $N \subset \mathbf{R}^3$ is a neighborhood of the box $[p,q] = \{x \in \mathbf{R}^3 : p_i \le x_i \le q_i, i = 1,2,3\}$ with equilibria only at p and q. F is said to be cooperative in $[p,q]$ if the Jacobian matrix $DF(x)$ at every $x \in [p,q]$ has nonnegative off-diagonal entries. In an earlier work [1], the author has proved that if F is cooperative and irreducible (that is, $DF(x)$ is irreducible for every $x \in [p,q]$) then every trajectory in $[p,q]$ converges (see [1, Lemma 1]). This result plays an important part in proving the existence of an orbit connecting p and q by Ważewski retraction method (see [1]). More general result than [1, Lemma 1] was proved by Hirsch (see [4, Thm. 10]). The proof of above results depends strongly on the irreducibility of $DF(x)$ (or the strong monotonicity of the flow generated by F). Using the idea present in [2], we can drop the condition that $DF(x)$ is irreducible for every $x \in [p,q]$. To be more precise, we shall prove the following:

Theorem 1 *Let F be a 3-dimensional C^1 vector field defined on a neighborhood N of the box $[p,q]$ with equilibria only at p and q. Suppose the off-diagonal terms in the Jacobian matrix $DF(x)$ are nonnegative for every $x \in [p,q]$. Then: either every trajectory in $[p,q]-Bd[p,q]$ approaches q, or else every trajectory in $[p,q]-Bd[p,q]$ approaches p, where Bd indicates the boundary of a set.*

By this theorem, we can improve [1, Thm. 2] which gives the existence of connecting orbits.

Corollary 2 *Suppose that the assumptions of Theorem 1 hold and that F points into the interior of $[p,q]$ on $Bd[p,q] - \{p,q\}$. Then there is an orbit of F contained in $[p,q]$ which joins p and q.*

Remark For an n-dimensional cooperative system, in order to guarantee the existence of an orbit connecting equilibria p and $q (p < q)$, one always imposes the condition that $DF(x)$

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Proof Using the method developed in [2] with minor modifications, we give the following proof. We only prove (ii) with $i = 1, j = 2, k = 3$. The other cases being similar.

It follows from the Kamke's theorem that $p \leq x(t) \leq q$ for all $t \geq 0$, therefore, $\omega(x) \subset [p,q]$ for any $x \in [p,q]$. By invariance, $y(t) \in \omega(x) \subset \mathrm{Bd}[p,q]$ for all $t \in \mathbf{R}$, where $y(t)$ is the solution of (1) passing through y. Since $y \in q - \partial\mathbf{R}^3_+$ and $y^i < q^i$ for $i = 1, 2, y^3 = q^3$. Therefor, $y(t) \in q - \mathrm{Int}(\pi^3)$ for $|t|$ sufficiently small where Int indicates the interior of a set. Set

$$t^* = \sup\{t : y^3(\tau) = q^3 \text{ for all } \tau \in [0,t]\}.$$

Obviously, $t^* > 0$. We claim that $t^* = \infty$. If not, $t^* < \infty$. By the definition of t^* and the continuity of $y(t), y^3(t) = q^3$ for all $t \in [0, t^*]$ and either $y^1(t^*) = q^1$ or $y^2(t^*) = q^2$. Without loss of generality, we suppose the latter case occurs. Denote $F = (F_1, F_2, F_3)$. Then $(y^1(t), y^2(t))$ for $t \in [0, t^*]$ is a solution of the following 2- dimensional cooperative system:

$$\dot{x} = F_1(x, y, q^3), \quad \dot{y} = F_2(x, y, q^3). \tag{2}$$

Since $q = (q^1, q^2, q^3)$ is an equilibrium of (1), $(x, y) = (q^1, q^2)$ is also a solution of (2). It follows from $q^1 > y^1, q^2 > y^2$ and the Kamke's theorem that $q^1 > y^1(t^*)$ and $q^2 > y^2(t^*)$, contradicting $y^2(t^*) = q^2$. This contradiction shows $t^* = \infty$, that is, $y(t) \in q - \pi^3$ for $t \geq 0$. If there is a $t_1 < 0$ such that $y(t_1) \notin q - \pi^3$, then it follows from $y(t) \in \mathrm{Bd}[p,q]$ for all $t \in \mathbf{R}$ that either $y(t_1) \in q - \mathrm{Int}(\pi^i)$ for $i \neq 3$ or $y(t_1) \in p + \mathrm{Int}(\pi^j)$ for some j. Then, as just proved, either $y(t) \in q - \pi^i$ for $i \neq 3$ or $y(t) \in p + \pi^j$ for some j. Then, as just proved, either $y(t) \in q - \pi^i$ for $i \neq 3$ or $y(t) \in P + \pi^j$ for some j here $t \geq t_1$, which contradicts $y \in q - \mathrm{Int}(\pi^3)$. This proves (ii) of lemma 3.

3. Proof of the Results

Proof of Theorem 1 By the Kamke's theorem, $\omega(x) \subset [p,q]$ for every $x \in [p,q]$. We claim that $\omega(x)$ contains an equilibrium. Otherwise, $\omega(x)$ is a cycle (see [6,Thm. 4.1]). By (ii) of Lemma 1, $[p,q]$ contains an equilibrium which is unrelated to any point of $\omega(x)$ by $>$ or $<$. So this equilibrium is either p or q and $\omega(x) \subset \mathrm{Bd}[p,q]$. Because $\omega(x)$ is a cycle, there is a point $y = (y^1, y^2, y^3) \in \omega(x)$ such that either $y^i - p^i > 0$ for two indices i or $y^j - q^j < 0$ for two indices j. By Lemma 3, either $\omega(x) \subset p + \pi^j$ for some j or $\omega(x) \subset q - \pi^k$ for some k, that is, $\omega(x)$ is a cycle of a 2-dimensional cooperative system, contradicting Lemma 2. This shows that our claim holds. Since $[p,q]$ only contains the equilibria p and q, without loss of generality, we can assume that $q \in \omega(x)$. If $\omega(x) - q \neq \emptyset$, then we can prove there is a point $y \in \omega(x)$ such that $y^i - q^i < 0$ for two indices i. By (ii) of Lemma 3, $y(t) \in q - \pi^j$ for some j and all $t \in \mathbf{R}$, that is, $y(t)$ is a solution of a 2-dimensional cooperative system. The compactness of $\omega(x)$ implies that $y(t)$ is bounded on $\mathbf{R}$. Applying lemma 2, we know that $y^i(t)$ is bounded and monotone for $|t|$ sufficiently large and for $i = 1, 2, 3$. Since $y(t) \in q - \pi^j$ for all $t \in \mathbf{R}$ and $q - \pi^j$ only contains the equilibrium q, $\lim_{t \to \infty} y(t) = q$ and $\lim_{t \to -\infty} y(t) = q$. So there exists a real number $T > 0$ such that $y(-T) \gneqq y$. By the Kamke's theorem,

$$q \gneqq y(-T) \gneqq y(T) \gneqq \cdots \gneqq y(nT).$$

It follows that $q = \lim_{n\to\infty} y(nT) \lneqq y \lneqq q$, a contradiction. This proves $\omega(x) = q$. So far we have proved that every trajectory in $[p, q]$ converges to p or q.

Let L denote the open line segment with endpoints p and q. If $x \in L$ and $\omega(x) = p$, that is, $\lim_{t\to\infty} x(t) = p$. For $z \in [p, x]$, applying the Kamke's theorem, we can conclude that $p \le z(t) \le x(t)$ for all $t \ge 0$, which implies that $z(t) \to p$ as $t \to \infty$. Therefor, every trajectory in $[p, x]$ converges to p. Similarly, if $x \in L$ and $\omega(x) = q$, then every trajectory in $[x, q]$ converges to q. Let $A = \{x \in L : \omega(x) = p\}$ and $B = \{x \in L : \omega(x) = q\}$. By the facts proved above, A and B are open in L with the relative topology, and $L = A \cup B$. From the connectedness of L we conclude that either $L = A$ or $L = B$. Without loss of generality, we may assume that $L = B$, that is, $\omega(x) = q$ for all $x \in L$. Fix any point $z \in [[p, q]$. Then there exists $x \in L$ with $x < z \le q$. Applying the Kamke's theorem, we know that $x(t) < z(t) \le q$ for all $t \ge 0$. Therefore, $\omega(x) = q$ implies that $\omega(z) = q$, that is, $\omega(z) = q$ for any $z \in [[p, q]$. This completes the proof.

Proof of Corollary 2 Since F points into the interior of $[p, q]$ on $\mathrm{Bd}[p, q] - \{p, q\}$, every point in $\mathrm{Bd}[p, q] - \{p, q\}$ is strict ingress. From Theorem 1 it follows that either every trajectory in $[p, q] - \{p\}$ approaches q; or else every trajectory in $[p, q] - \{q\}$ approaches p. The rest proof is quite the same as that of [1,Thm. 2]. So we omit it.

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三维长方体$[p,q]$内的合作系统

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摘　　要

设F是三维长方体$[p,q]$内的合作向量场，且在此长方体内F的奇点仅为p和q. 我们证明了$[p,q]$内的每条正半轨线或者收敛于p，或者收敛于q. 此外，如果在$[p,q]$的边界上除p和q两点外，F的方向均指向$[p,q]$的内部，则$[p,q]$内存在唯一的轨道连结p和q.