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The Joint Distribution of kth Record Times and kth Record values*

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Abstract In this paper we give the joint distribution of kth record times and kth record values with direct method. And then some interesting relations can be derived.

Let $\{X_n; n \geq 1\}$ be an independent identically distributed (i.i.d.) sequence of random variables (r.v.'s) with absolutely continuous distribution function $F(x) = P(X_1 < x)$ and density function $f(x)$. For $n = 1, 2, \ldots$, we denote the order statistics of $X_1, X_2, \cdots, X_n$ by $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$ and define the kth record times as follows:

$$U^{(k)}(0) = 0, U^{(k)}(1) = k$$

and

$$U^{(k)}(n+1) = \min\{j : j > U^{(k)}(n), X_j > X_{j-k,j-1}\}(n \geq 1, k \geq 1).$$

The quantities $\Delta^{(k)}(n) = U^{(k)}(n) - U^{(k)}(n-1)(n \geq 2)$ are called the kth inter-record times and the sequence of r.v.'s $X^{(k)}(n) = X_{U^{(k)}(n)-k+1, U^{(k)}(n)}, n \geq 1$, the sequence of kth record values. For $k = 1$ we obtain the usual record, inter-record times and record values. A large number of publications has been devoted to records, we refer to may be found in [1], [2] and [3] for a surrey. In this paper we shall give the joint distribution of $\{U^{(k)}(i), X^{(k)}(i); 1 \leq i \leq n\}$ by proving the following

Theorem *For any $n \geq 1$, the joint density function of $\bigcap_{i=1}^{n}\{U^{(k)}(i), X^{(k)}(i)\}$ is*

(1)
$$f_{(\bigcap_{i=1}^{n}\{U^{(k)}(i), X^{(k)}(i)\})}(j_1, x_1; j_2, x_2; \cdots; j_n, x_n)$$
$$= k^n (1 - F(x_n))^{k-1} f(x_n) \prod_{i=1}^{n-1} [f(x_i) F^{j_{i+1} - j_i - 1}(x_i)],$$

where $j_1 = k < j_2 < \cdots < j_n$, $x_i (1 \leq i \leq n)$ are real and $x_1 < x_2 < \cdots < x_n$.

Proof Let

$$D_1 = \{X_1, X_2, \cdots, X_k\} = \{X_1^{(1)}, X_2^{(1)}, \cdots, X_k^{(1)}\} = \{X_{1,k}^{(1)}, X_{2,k}^{(1)}, \cdots, X_{k,k}^{(1)}\};$$
$$D_i = \{X_1^i, X_2^i, \cdots, X_k^i\} = \{X_{j_i}\} \bigcup \{X_{2,k}^{(i-1)}, X_{3,k}^{(i-1)}, \cdots, X_{k,k}^{(i-1)}\}(2 \leq i \leq n),$$

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then for arbitrarily small positive number group $\{\delta_i, 1 \le i \le n\}$, from to the definition of $U^{(k)}$ and $X^{(k)}(n)$, as well as independence of $\{X_n\}$, we have

$$
\begin{aligned}
P(\bigcap_{i=1}^{n}\{U^{(k)}(i) &= j_i, x_i < X^{(k)}(i) < x_i + \delta_i\}) \\
&= P(\bigcap_{i=1}^{n}\{\text{ one element of the } D_i \text{ in } (x_i, x_i + \delta_i)\} \\
&\quad \bigcap\{\text{ other elements of } D_n \text{ are larger then } x_n + \delta_n\} \\
&\quad \bigcap\bigcap_{i=1}^{n-1}\{\max\{X_{j_i+1}, \cdots, X_{j_{i+1}-1}\} \le x_i\}) + O(\max_{1\le i\le n}\{\delta_i^{n+1}\}) \\
&= k^n \prod_{i=1}^{n}[F(x_i+\delta_i) - F(x_i)] \cdot [1 - F(x_n)]^{k-1} \cdot \prod_{i=1}^{n-1} F^{j_{i+1}-j_i-1}(x_i) \\
&\quad + O(\max_{1\le i\le n}\{\delta_i^{n+1}\})
\end{aligned}
$$

where $O(\max_{1\le i\le n}\{\delta_i^{n+1}\})$ means terms of order $\max_{1\le i\le n}\{\delta_i^{n+1}\}$ and includes the probability of realizations of $x_i < X^{(k)}(i) \le x_i + \delta_i, 1 \le i \le n$ in which more than one element of some D_i in $(x_i, x_i + \delta_i), 1 \le i \le n$. Dividing both sides by $\prod_{i=1}^{n} \delta_i$ and letting $\max_{1\le i\le n}\{\delta_i\} \to 0$, (1) is obtained.

Remark 1 This theorem was proved for $k = 1$ and $F(x) = x$ by Renyi ([4]).

Corollary 1 *The sequence of vectors* $\{(U^{(k)}(n), X^{(k)}(n)), n \ge 1\}$ *forms a homogeneous Markov chain with transition density function*

(2)
$$
\begin{aligned}
&f_{(U^{(k)}(n), X^{(k)}(n))}(j, y \mid U^{(k)}(n-1) = i, X^{(k)}(n-1) = x) \\
&= k \cdot f(y) F^{j-i-1}(x) \left[\frac{1 - F(y)}{1 - F(x)}\right]^{k-1}.
\end{aligned}
$$

Proof From (1) we get

$$
\begin{aligned}
&f_{(U^{(k)}(n), X^{(k)}(n))}(j_n, x_n \mid \bigcap_{i=1}^{n-1}\{U^{(k)}(i) = j_i, X^{(k)}(i) = x_i\}) \\
&= \frac{f_{(\bigcap_{i=1}^{n} U^{(k)}(i), X^{(k)}(i)\})}(j_1, x_1; j_2, x_2; \cdots; j_n, x_n)}{f_{(\bigcap_{i=1}^{n-1} U^{(k)}(i), X^{(k)}(i)\})}(j_1, x_1; j_2, x_2; \cdots; j_{n-1}, x_{n-1})} \\
&= k f(x_n) F^{j_n - j_{n-1} - 1}(x_{n-1}) \left[\frac{1 - F(x_n)}{1 - F(x_{n-1})}\right]^{k-1} \\
&= f_{(U^{(k)}(n), X^{(k)}(n))}(j_n, x_n \mid U^{(k)}(n-1) = j_{n-1}, X^{(k)}(n-1) = x_{n-1}).
\end{aligned}
$$

This is (2).

Corollary 2 *$X^{(k)}(n)$ is independent of $\{U^{(k)}(1), U^{(k)}(2), \cdots, U^{(k)}(n-1)\}$ and $\{X^{(k)}(n), n \geq 1\}$ is a homogeneous Markov chain with transition probabilities*

$$(3).\qquad P(X^{(k)}(n) \geq y \mid X^{(k)}(n-1) = x) = \left[\frac{1-F(y)}{1-F(x)}\right]^k$$

Proof (3) can be got from (2) after some calculations

$$f_{X^{(k)}(n)}(y \mid \{U^{(k)}(n-1) = i, X^{(k)}(n-1) = x\} \bigcap \bigcap_{i=1}^{n-2} \{U^{(k)}(i) = j_i, X^{(k)}(i) = x_i\})$$
$$= \sum_{j=i+1}^{\infty} f_{(U^{(k)}(n), X^{(k)}(n))}(j, y \mid U^{(k)}(n-1) = i, X^{(k)}(n-1) = x)$$
$$= k \cdot F(y) \left[\frac{1-F(y)}{1-F(x)}\right]^{k-1} \sum_{j=0}^{\infty} F^j(x) = \frac{k \cdot f(y)(1-F(y))^{k-1}}{1-F(x))^k}$$
$$= f_{X^{(k)}(n)}(y \mid X^{(k)}(n-1) = x).$$

Remark 2 Deheuvels ([5]) found the equivalence relation between $\{X^{(k)}(n), n \geq 1\}$ and the order statistics $\{X_{n,N}, 1 \leq n \leq N\}$ of $\{X_1, X_2, \cdots, X_N\}$ and gave (3) in a different way.

Remark 3 From (3), we can directly get the formula of the distribution of $X^{(k)}(n)$.

Corollary 3 *$U^{(k)}(n)$ and $X^{(k)}(n)$ are conditional independent under $U^{(k)}(n-1) = i$ and $X^{(k)}(n-1) = x$*

Proof It follows from (2) that

$$(4)\qquad P(U^{(k)}(n) = j \mid U^{(k)}(n-1) = i, X^{(k)}(n-1) = x),$$
$$= \int_x^{\infty} f_{(U^{(k)}(n), X^{(k)}(n))}(j, y \mid U^{(k)}(n-1) = i, X^{(k)}(n-1) = x) dy$$
$$= F^{j-i-1}(x) \cdot (1 - F(x)).$$

In accordance with (2), (3) we have

$$(5)\qquad f_{(U^{(k)}(n), X^{(k)}(n))}(j, y \mid U^{(k)}(n-1) = i, X^{(k)}(n-1) = x)$$
$$= f_{X^{(k)}(n)}(y \mid X^{(k)}(n-1) = x) \cdot P(U^{(k)}(n) = j \mid U^{(k)}(n-1) = i, X^{(k)}(n-1) = x).$$

Corollary 4 *$\{U^{(k)}(n), n \geq 1\}$ is a homogeneous Markov chain with transition probabilities*

$$(6)\qquad P(U^{(k)}(n) = j \mid U^{(k)}(n-1) = i) = \frac{k}{j-k} \cdot \frac{B(j-k+1, k)}{B(i-k+1, k)} \text{ for any } j > i \geq k \geq 1,$$

where B-function is defined by $B(p, q) = \int_0^1 u^{p-1}(1-u)^{q-1} du$ for $Re\ p > 0$ and $Re\ q > 0$.

Proof From (1), we have

$$
\begin{aligned}
&P(\bigcap_{i=1}^{n}\{U^{(k)}(i)=j_i\})\\
&=k^n\int_{-\infty}^{+\infty}(1-F(x_n))^{k-1}dF(x_n)\int_{-\infty}^{x_n}F^{j_n-j_{n-1}-1}(x_{n-1})dF(x_{n-1})\\
&\quad\cdots\int_{-\infty}^{x_2}F^{j_2-k-1}(x_1)dF(x_1)=\frac{k^n}{\prod_{i=1}^{n}(j_i-k)}\int_0^1 v^{j_n-k}(1-v)^{k-1}dv\\
&=\frac{k^n}{\prod_{i=1}^{n}(j_i-k)}B(j_n-k+1,k),
\end{aligned}
$$

so we have

$$
P(U^{(k)}(n)=j_n\mid\bigcap_{i=1}^{n-1}\{U^{(k)}(i)=j_i\})=\frac{k}{j_n-k}\cdot\frac{B(j_n-k+1,k)}{B(j_{n-1}-k+1,k)},
$$

(6) is obtained.

Remark 4 The result for k=1 was shown by Renyi ([4]).

Corollary 5 $\{\Delta^{(k)}(n),X^{(k)}(n);n\geq 1\}$ *forms a homogeneous Markov chain with transition density function*

$$
\begin{aligned}
(7)\qquad &f_{(\Delta^{(k)}(n),X^{(k)}(n))}(j,y\mid\Delta^{(k)}(n-1)=i,X^{(k)}(n-1)=x)\\
&=kf(y)F^{j-1}(x)\left(\frac{1-F(y)}{1-F(x)}\right)^{k-1}
\end{aligned}
$$

Proof From (2) and the definition of $\Delta^{(k)}(n)$, we can get

$$
\begin{aligned}
&f_{(\Delta^{(k)}(n),X^{(k)}(n))}(j_n,x_n\mid\bigcap_{i=1}^{n-1}\{\Delta^{(k)}(i)=j_i,X^{(k)}(i)=x_i\})\\
&=f_{(U^{(k)}(n),X^{(k)}(n))}(\sum_{i=1}^{n}j_i,x_n\mid U^{(k)}(n-1)\sum_{i=1}^{n-1}j_i,X^{(k)}(n-1)=x_{n-1}\\
&=kf(x_n)F^{j_n-1}(x_{n-1})\left[\frac{1-F(x_n)}{1-F(x_{n-1})}\right]^{k-1},
\end{aligned}
$$

this is (7).

Remark 5 From (7) we can see that $\{\Delta^{(k)}(i),1\leq i\leq n\}$ are independent if $\{X^{(k)}(i),1\leq i\leq n-1\}$ are fixed. And this property leads to the expression for the distribution of $\Delta^{(k)}(n)$ for any $k\geq 1$ and $n\geq 1$(see [2]).

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References

[1] J. Galambos, *The Asymptotic Theory of Extreme Order Statistics,* John Wiley, New York, 1978.

[2] V.B. Nevzorov, *Records*, Theory Probab. Appl. **32**(1987),**4**,201-228.

[3] S.I. Resnick, *Extreme Values, Regular Variation and Point Processes,* Springer, New York, 1987.

[4] A. Renyi, *On the extreme elements of observations,* in Selected Paper of Alfred Renyi, Adakemiai Kiado, **3**(1976), 50-65.

[5] P. Deheuvels, *The characterization of distributions by order statistics and record values—a unified approach,* J. Appl. Prob. **21**(1984), 326-334.

关于k阶记录时间与k阶记录值的联合分布

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摘　　要

设独立同分布随机变量序列$\{x_n;n\geqslant 1\}$的分布函数$F(x)=p(x_1<x)$绝对连续,$\{U^{(k)}(n);n\geqslant 1\}$,$\{X^{(k)}(n);n\geqslant 1\}$分别为$\{x_n;n\geqslant 1\}$的$k$阶记录时间序列和$k$阶记录值序列.本文我们用直接方法求出了$\{U^{(k)}(i),X^{(k)}(i);1\leqslant i\leqslant n\}$的联合分布,从而证明了$k$阶记录时间序列及$k$阶记录值序列的马氏性,并导出了它们之间的一些关系,推广了关于一阶记录时间及记录值的结果.