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A Note on Stochastic Ordering ($\leq_d$)*

Hu Nanhua
(Dept. Math., Xiamen University,)

Abstract In discussing the convergence in probability (in distribution) of a sequence of random variables, it is often used that if for any $n, P\{X_n \leq Y_n \leq Z_n\} = 1$ and $X_n \xrightarrow{P(d)} Y, Z_n \xrightarrow{P(d)} Y$, then $Y_n \xrightarrow{P(d)} Y$. It is shown now that the stochastic ordering condition $X_n - Y \leq_d Y_n - Y \leq_d X_n - Y (X_n \leq_d Y_n \leq_d Z_n)$ is a more general dominating condition than $P\{X_n \leq Y_n \leq Z_n\} = 1$ in ensuring the convergence in probability (in distribution) of $\{Y_n\}$.

It is well known ([1], [2], [3]) that the comparison methods of stochastic ordering have been widely used in the fields of queueing theory, reliability and stochastic point processes. It is shown now that even in a primary subject of probability theory – convergence in probability (in distribution), the stochastic ordering ($\leq_d$) still has its particular "dominating power".

Definition 1[1] *Let X and Y be two random variables on a probability space $(\Omega, \mathcal{F}, P)$, F_x and F_y their respective distribution functions. It is called that according to the stochastic ordering ($\leq_d$), random variable X is "smaller" than Y if $P\{X \leq a\} = F_x(a) \geq F_y(a) = P\{Y \leq a\}$ for any real number a. In this case we write $X \leq_d Y$ or $F_x \leq_d F_y$.*

Similarly to the comparison test of the convergences of series and sequence in mathematical analysis, when discussing the convergence of a sequence of random variables in probability theory, we have the following

Proposition 2 *Let $\{X_n\}, \{Y_n\}$ and $\{Z_n\}$ be sequences of random variables on $(\Omega, \mathcal{F}, P)$. If for any $n, P\{X_n \leq Y_n \leq Z_n\} = 1$ and $X_n \xrightarrow{P(d)} Y, Z_n \xrightarrow{P(d)} Y$, then $Y_n \xrightarrow{P(d)} Y$.*

Obviously, we need only to consider the case of $Y = 0$ when proving this proposition in the case of convergence in probability.

After introducing the concept of $\leq_d$, we find that the condition $P\{X_n \leq Y_n \leq Z_n\} = 1$ may be generalized to $X_n \leq_d Y_n \leq_d Z_n$ in getting the conclusion of Proposition 2. For this purpose, we need the following two propositions and one theorem.

Proposition 3 *Let X and Y be two random variables on $(\Omega, \mathcal{F}, P)$. If $P\{X \leq Y\} = 1$, then $X \leq_d Y$. The inverse is not true.*

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Proof If $P\{X \le Y\} = 1$, then for any real number a, we have

$$F_x(a) = P\{X \le a\} \ge P\{Y \le a\} = F_y(a).$$

It is just $X \le_d Y$. On the other hand, let $(\Omega, \mathcal{F}, P) = ((0,1], \mathcal{B}(0,1], \mu(0,1])$, where $\mu(0,1]$ is the Lebesgue measure on $(0,1]$. For $k = 0,1,2,\ldots$, let $X(\omega) = -k$ if $\omega \in ((1/2)^{k+1}, (1/2)^k], Y(\omega) = k-1$ if $\omega \in ((2/3)^{k+1}, (2/3)^k]$. Then for any real number a,

$$\begin{aligned} &F_y(a) = 0 < F_x(a), && a < -1, \\ &F_y(a) = 1/3 < 1/2 = F_x(a), && \text{if } -1 \le a < 0, \quad \text{i.e. } X \le_d Y, \\ &F_y(a) < 1 = F_x(a), && a \ge 0, \end{aligned}$$

but $P\{X \le Y\} = \mu(0, 2/3] = 2/3 < 1$. □

Proposition 4 *Let $\{X_n\}, \{Y_n\}$, and $\{Z_n\}$ be three sequences of random varibales on $(\Omega, \mathcal{F}, P)$. If for any $n, X_n \le dY_n \le dZ_n$ and $X_n \xrightarrow{P} 0, Z_n \xrightarrow{P} 0$, then $Y_n \xrightarrow{P} 0$.*

Proof For any positive number ε, from

$$\begin{aligned} P\{|Y_n| \ge \varepsilon\} &= P\{Y_n \ge \varepsilon\} + P\{Y_n \le -\varepsilon\} \le P\{Y_n > \varepsilon/2\} + P\{Y_n \le -\varepsilon\} \\ &\le P\{X_n > \varepsilon/2\} + P\{Z_n \le -\varepsilon\} \le P\{|X_n| \ge \varepsilon/2\} + P\{|Z_n| \ge \varepsilon\}, \end{aligned}$$

and $X_n \xrightarrow{P} 0, Z_n \xrightarrow{P} 0$, we obtain the conclusion. □

Theorem 5 *Let $\{X_n\}, \{Y_n\}$ and $\{Z_n\}$ be three sequences of random variables and Y a random variable on $(\Omega, \mathcal{F}, P)$. If $X_n \xrightarrow{d} Y, Z_n \xrightarrow{d} Y$, and $X_n \le_d Y_n \le_d Z_n$ for any n, then $Y_n \xrightarrow{d} Y$.*

Proof Let the distribution functions of X_n, Y_n, Z_n and Y be F_n, G_n, H_n and G, respectively, C the continuity set of G and x_0 any fixed point in C. After taking any a in C such that $a > x_0$, we have

$$P\{Y_n \le x_0\} \le P\{Y_n \le a\} \le P\{X_n \le a\} \text{ or } G_n(x_0) \le F_n(a).$$

Therefore

$$\overline{\lim} G_n(x_0) \le \overline{\lim} F_n(a) = G(a). \tag{1}$$

Because x_0 is in C, we have $\overline{\lim} G_n(x_0) \le G(x_0)$ while a, in (1), strictly decreases and converges to x_0.

Similarly, $\underline{\lim} G_n(x_0) \ge G(x_0)$, and $\lim G_n(x_0) = G(x_0)$. □

From Proposition 4, we can easily obtain

Theorem 6 *Let $\{X_n\}, \{Y_n\}$ and $\{Z_n\}$ be three sequences of random variables and Y a random variable on $(\Omega, \mathcal{F}, P)$. If for any $n, X_n - Y \le_d Y_n - Y \le_d Z_n - Y$ and $X_n \xrightarrow{P} Y, Z_n \xrightarrow{P} Y$, then $Y_n \xrightarrow{P} Y$.*

Besides this, from the reflecxivity of stochastic ordering, we can get another equivalent definition of the convergence in probability (in distribution).

Corollary 7 *Let $\{Y_n\}$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$. The necessary and sufficient condition for $Y_n \stackrel{P(d)}{\rightarrow} Y$ is that there exist two sequences of random variables $\{X_n\}, \{Z_n\}$, such that $X_n \stackrel{P(d)}{\rightarrow} Y, Z_n \stackrel{P(d)}{\rightarrow} Y$, and $X_n - Y \leq_d Y_n - Y \leq_d Z_n - Y (X_n \leq_d Y_n \leq_d Z_n)$ for any n.*

Proof The necessity follows at once after taking $X_n = Z_n = Y_n$.

References

[1] D. Stoyan, *Comparison Methods for Queues and other Stochastic Models,* Wiley, New York, 1983.

[2] R.E. Barlow and F. Proschan, *Mathematical Theory of Reliability,* Wiley, New York, 1965.

[3] Y.L. Deng, *On the comparison of point processes,* J. Appl. Prob., **22**(1985), 300-313.

关于随机序的一个注记

胡南桦

(厦门大学数学系,361005)

摘　　要

本文引入随机序($\leq_d$)的概念,说明在讨论随机变量列的依概率(依分布)收敛问题时,它是一个颇为恰当的"控制尺度".