Journal of Mathematical Research & Exposition
Vol.13, No.1, Feb. 1993

# The $L_p$-saturation of Mixed Exponential Type Integral Operators*

*Chen Wenzhong* *Cui Zhenlu*
(Dept. of Math., Xiamen University)

**Abstract.** We study the properties of the mixed exponential type integral operators in $L_p$−space and established their $L_p$−saturation theorems.

## 1. Introduction

Let $I=[0,1]$ or $[0,+\infty)$ and $n^+$ denote $n$ or $+\infty$. The sequence $\{\lambda_{nk}(x)\}_{k=0}^{n^+}$ is called an exponential type kernel on $I$ and denoted by $\{\lambda_{nk}\}_{k=0}^{n^+}\in E_\alpha$, if it satisfies the following conditions:

i) For $x\in I^0, \lambda_{nk}(x)>0$ $(0\le k\le n^+)$ and $\sum_{k=0}^{n^+}\lambda_{nk}(x)=1$, where $I^0$ is the interior of $I$.

ii) $\lambda_{nk}\in C^\infty(I)$ $(0\le k\le n^+)$ and satisfies

$$x(1+\alpha x)\lambda'_{nk}(x)=(k-nx)\lambda_{nk}(x), \tag{1.1}$$

where $\alpha=-1$ or $\alpha\ge 0$ such that $x(1+\alpha x)\ge 0$ $(x\in I^0)$.

By computation, we obtain

i) If $\alpha=-1$, then $I=[0,1], n^+=n$ and $\lambda_{nk}(x)=p_{nk}(x)=\binom{n}{k}x^k(1-x)^{n-k}$;

ii) If $\alpha\ge 0$, then $I=[0,+\infty), n^+=+\infty$ and

$$\lambda_{nk}=\begin{cases} e^{-nx}\frac{(nx)^k}{k!}, & \alpha=0,\\ \frac{\Gamma(\frac{n}{\alpha}+k)}{k!\Gamma(\frac{n}{\alpha})}\alpha^k x^k(1+\alpha x)^{-\frac{n}{\alpha}-k}, & \alpha>0.\end{cases}$$

Hence, if $\{\lambda_{nk}\}_{k=0}^{+}\in E_\alpha$, then

$$\begin{aligned} &\text{i) } \textstyle\sum_{k=0}^{n^+}\frac{k}{n}\lambda_{nk}(x)=x;\\ &\text{ii) } \textstyle\sum_{k=0}^{n^+}(\frac{k}{n})^2\lambda_{nk}(x)=x^2+\frac{x(1+\alpha x)}{n};\\ &\text{iii) } (n-\alpha)\textstyle\int_I t^j\lambda_{nk}(t)\,dt=\frac{(k+j)!\Gamma(\frac{n}{\alpha}-j-1)}{k!\alpha^j\Gamma(\frac{n}{\alpha}-1)}, j=0,1,2.\end{aligned} \tag{1.2}$$

*Received Nov. 28, 1990. Supported by Fujian Natural Science Foundation.

Let $\{\lambda_{nk}\}_{k=0}^{n^+} \in E_\alpha$, for $f \in L_p(I), 1 \le p \le +\infty$, setting

$$\varphi_{nk}(f) = (n-\alpha)\int_I f(t)\lambda_{nk}(t)\,dt,$$

$$T_n(f,x) = \sum_{k=0}^{n^+} \lambda_{nk}(x)\varphi_{nk}(f).$$

We call $T_n$ the mixed exponential type integral operators. In [1] we studied the approximation theorem of $T_n$ in $C(I)$. In this paper we will establish $L_p$−saturation theorem for $T_n$.

## 2. Some lemmas

Let $m \in N$ and $\{\lambda_{nk}\}_{k=0}^{n^+} \in E_\alpha$. Setting

$$R_{nm}(x) = \sum_{k=0}^{n^+} \lambda_{nk}(x)(n-\alpha)\int_I (t-x)^m \lambda_{nk}(t)\,dt,$$

we have

**Lemma 1**[1] *For $m \in N$ and $x \in I$ there holds*

$$\begin{aligned}&(n-(m+2)\alpha)R_{n,m+1}(x)\\&= x(1+\alpha x)R'_{n,m}(x) + 2mx(1+\alpha x)R_{n,m+1}(x) + (m+1)(1+2\alpha x)R_{n,m}(x).\end{aligned}$$

We can easily verify

$$R_{n0}(x) = 1, \quad R_{n1}(x) = \frac{1+2\alpha x}{n-2\alpha},$$

$$R_{n2}(x) = \frac{2x(1+\alpha x)}{n-2\alpha} + o_x(\frac{1}{n^2}).$$

Generally, by Lemma 1 we have

**Corollary 1** *Let $m \in N$ and $x \in I$. Then*

$$R_{nm}(x) = o_x\left(\frac{1}{n^{[\frac{m+1}{2}]}}\right),$$

*and $n^{[\frac{m+1}{2}]}R_{n,m}(x)$ is uniformly bounded on any compact subset of $I$.*

Using Corollary 1 and Lemma 5.2 in [8] we can prove the following Voronovskaja formula for the operator $T_n$.

**Lemma 2** *Let $f \in L_p(I) \cap C^2(I)$. Then*

$$\lim_{n\to\infty} n(T_n(f,x) - f(x)) = (x(1+\alpha x)f'(x))',$$

*and the limit holds uniformly on any compact subset of $I$.*

**Lemma 3** *Let $\{L_n\}_{n\in N}$ be a positive linear operator sequence from $C(I)$ to $C(I)$, $e_k(x)=x^k, k=0,1,2$ and $\mu_n^2=\max_{k=0,1,2}\| L_n(e_k)-e_k \|_\infty$. For any $g\in C^2(I)$, we have*

$$\| L_n(g)-g \|_\infty \le K\mu_k^2(\| g \|_\infty+\| g'' \|_\infty). \tag{2.1}$$

**Proof** By Taylor's formula

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t (t-u)g''(u)\,du,$$

we have

$$\begin{aligned}|L_n(g,x)-g(x)| &\le |g(x)|\,|L_n(1,x)-1|+|g'(x)|\,|L_n(t-x,x)|\\ &+ \| g'' \|_\infty \frac{1}{2}L_n((t-x)^2,x).\end{aligned}$$

Using Stein inequality, we establish (2.1).

Let $C_0^2(I)=\{\psi \mid \psi\in C^2(I)\ \ \overline{\mathrm{supp}\psi}\subset I^0\}$, $A_n(f,\psi)=n\int_I(T_n(f,x)-f(x))\psi(x)\,dx$. Then we have

**Lemma 4** *For $\psi\in C_0^2(I)$, $f\in L_p(I)$ there exists $M_\psi>0$ satisfying*

$$|A_n(f,\psi)|\le M_\psi\| f \|_p.$$

**Proof** Let $\psi\in C_0^2(I)$, using Lemma 3, we have

$$\| T_n(\psi)-\psi \|_\infty \le \frac{K}{n}(\| \psi \|_\infty+\| \psi'' \|_\infty),$$

where $K>0$ is a constant.

By computation it is easy to verify

$$\int_I T_n(f,x)\psi(x)\,dx=\int_I T_n(\psi,x)f(x)\,dx$$

hence we have

$$\begin{aligned}|A_n(f,\psi)| &= n\mid \int_I T_n(f,x)-f(x))\psi(x)\,dx \mid\\ &= n\mid \int_I T_n(\psi,x)-\psi(x))f(x)\,dx \mid\\ &\le n\| T_n(\psi)-\psi \|_\infty\| f \|_p\\ &\le K(\| \psi \|_\infty+\| \psi'' \|_\infty)\| f \|_p.\end{aligned}$$

Setting $\varphi(x)=x(1+\alpha x)$, $U_p=\{g\mid g\in L_p(I),\varphi g''\in L_p(I)\}$, we have the following lemma (cf. [7]):

**Lemma 5** *Let $g \in U_p$ $(1 \le p \le +\infty)$. Then*

*i)* $\| g' \|_p \le K(\| \varphi g'' \|_p + \| g \|_p)$;

*ii)* $\| (\varphi g')' \|_p \le K(\| \varphi g'' \|_p + \| g \|_p)$,

*where $K > 0$ is an absolute constant.*

Setting $V_n(f,x) = \sum_{k=0}^{n^+} f(\frac{k}{n})\lambda_{nk}(x)$. We have

**Lemma 6** *Let $g \in U_p, 1 < p \le +\infty$. Then*

$$\| V_n(g,\cdot) - g(\cdot) \|_p \le \frac{K}{n}(\| \varphi g'' \|_p + \| g \|_p).$$

**Proof** By Taylor's formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-\tau)g''(\tau)\,d\tau$$

and $V_n(t-x,x) = 0$, we obtain

$$|V_n(g,x) - g(x)| \le | V_n(\frac{|t-x|}{\varphi(x)} \int_x^t \varphi(\tau)g''(\tau)d\tau, x) | .$$

Using the maximal function $M(\cdot)$ and Hardy inequality, we obtain

$$\begin{aligned} \| V_n(g,\cdot) - g(\cdot) \|_p &\le \| V_n\left(\frac{(t-x)^2}{\varphi(\cdot)} M(\varphi g'',\cdot),\cdot\right) \|_p \\ &\le \frac{1}{n} \| M(\varphi g'',\cdot) \|_p \le \frac{K_p}{n} \| \varphi g'' \|_p \\ &\le \frac{K}{n}(\| \varphi g'' \|_p + \| g \|_p). \end{aligned}$$

**Lemma 7** *Let $f \in L_p(I)$, $1 \le p \le +\infty$. Then*

$$\| T_n f \|_p \le \| f \|_p, \quad n > \alpha. \tag{2.2}$$

**Proof** By the Riesz-Thorin theorem, we need only to show the cases (2.2) $p = 1$ and $p = +\infty$.

For $p = 1$,

$$\begin{aligned} \| T_n f \|_1 &= \int_I | \sum_{k=0}^{n^+} \lambda_{nk}(x)(n-\alpha) \int_I \lambda_{nk}(t) f(t)\,dt \mid dx \\ &\le \sum_{k=0}^{n^+} (n-\alpha) \int_I \lambda_{nk}(x)\,dx \int_I |f(t)\lambda_{nk}(t)\,dt \\ &= \| f \|_1. \end{aligned}$$

For $p = +\infty$,

$$\begin{aligned} \| T_n f \|_\infty &= \| \sum_{k=0}^{n^+} \lambda_{nk}(x)(n-\alpha) \int_I \lambda_{nk}(t)\,dt \| \\ &\leq \| f \|_\infty \sum_{k=0}^{n^+} (n-\alpha)\lambda_{nk}(x)(n-\alpha) \int_I \lambda_{nk}(t)\,dt \\ &= \| f \|_\infty, \end{aligned}$$

the lemma is proved.

**Lemma 8** *Let $g \in U_p$, $1 < p \leq +\infty$. Then*

$$\| T_n(g) - V_n(g) \|_p \leq \frac{K}{n}(\| \varphi g'' \|_p + \| g \|_p).$$

**Proof** $V_n(g,x) - T_n(g,x)$

$$\begin{aligned} &= \sum_{k=0}^{n^+} \lambda_{nk}(x)(n-\alpha) \int_I g(\frac{k}{n}) - g(t))\lambda_{nk}(t)\,dt \\ &= \sum_{k=0}^{n^+} \lambda_{nk}(x)(n-\alpha) \int_I (g'(t)(\frac{k}{n} - t) - \int_t^{\frac{k}{n}} (\frac{k}{n} - \tau) g''(\tau)\,d\tau)\lambda_{nk}(t)\,dt \\ &= \frac{1}{n} \sum_{k=0}^{n^+} \lambda_{nk}(x)(n-\alpha) \int_I \varphi(t) g'(t)\,d\lambda_{nk}(t) \\ &\quad + \sum_{k=0}^{n^+} \lambda_{nk}(x)(n-\alpha) \int_I \int_t^{\frac{k}{n}} (\frac{k}{n} - \tau) g''(\tau) d\tau \lambda_{nk}(t)\,dt \\ &\triangleq I_1 + I_2. \end{aligned}$$

Using Lemma 5 we obtain

$$\begin{aligned} \| I_1 \|_p &= \frac{1}{n} \| \sum_{k=0}^{n^+} \lambda_{nk}(x)(n-\alpha) \int_I (\varphi(t) g'(t))' \lambda_{nk}(t)\,dt \|_p \\ &\leq \frac{1}{n} \| \varphi g')' \|_p \leq \frac{K}{n}(\| \varphi g'' \|_p + \| g \|_p), \\ \| I_2 \|_p &= \| \sum_{k=0}^{n^+} \lambda_{nk}(x)(n-\alpha) \int_I \int_t^{\frac{k}{n}} (\frac{k}{n} - \tau) g''(\tau) d\lambda_{nk}(t)\,d\tau \|_p \\ &\leq \frac{1}{n} \| \sum_{k=0}^{n^+} \lambda_{nk}(x)(n-\alpha) \int_I |\varphi(t) g''(x)|' \lambda_{nk}(t)\,dt \|_p \\ &\leq \frac{1}{n} \| \varphi g'' \|_p \leq \frac{K}{n}(\| \varphi g'' \|_p + \| g \|_p). \end{aligned}$$

Hence we have

$$\begin{aligned}\| V_n(g,\cdot) - T_n(g,\cdot) \|_p &\leq \| I_1 \|_p + \| I_2 \|_p \\ &\leq \frac{K}{n}(\| \varphi g'' \|_p + \| g \|_p).\end{aligned}$$

By Lemma 6 and Lemma 8 we obtain

**Lemma 9** *Let $g \in U_p$, $1 < p \leq +\infty$. Then*

$$\| T_n(g) - g \|_p \leq \frac{K}{n}(\| \varphi g'' \|_p + \| g \|_p).$$

## 3 Theorems and their proofs

Setting $S_p = \{f \mid f \in L_p(I)$, and for $p > 1, \varphi(x)f''(x) \in L_p(I)$, for $p = 1, \varphi(x)f'(x) \in BV(I)\}$.

**Theorem 1** *Let $f \in L_p(I), 1 \leq p \leq +\infty$. Then*

$$\| T_n f - f \|_p = o(\frac{1}{n})$$

*iff $f \equiv constant(a.e.)$.*

**Proof** It is enough to prove the necessity part.

Let $\| T_n f - f \|_p = o(\frac{1}{n})$. Then

$$\begin{aligned}|A_n(f,\psi)| &\leq n \int_I |T_n(f,x) - f(x)|\, |\psi(x)|\, dx \\ &\leq \| \psi \|_\infty n \| T_n f - f \|_p,\end{aligned}$$

hence $\lim_{n\to\infty} A_n(f,\psi) = 0$, which implies the following homogeneous integral equation

$$\int_I (\varphi(x)\psi'(x))' f(x)\, dx = 0.$$

For $y \in I^0$, let

$$\psi_y(x) = \begin{cases} (x-y)^2, & 0 \leq x \leq y. \\ 0, & x \geq y, \end{cases}$$

then $\psi_y(x) \in C_0^2(I)$.

Hence for $y \in I^0$,

$$\int_0^y (\varphi(x)\psi_y'(x))' f(x)\, dx = 0,$$

and therefore $f(x) \equiv \mathit{constant}$ (a.e.).

**Theorem 2** *Let $f \in L_p(1), p \geq 1$. Then $\| T_n f - f \|_p = O(\frac{1}{n})$ implies $f \in S_p$.*

**Proof** Let $f \in C^2(I), \psi \in C_0^2(I)$. We have

$$\begin{aligned}\lim_{n\to\infty} A_n(f,\psi) &= \int_I (\varphi(x)f'(x))'\psi(x)\,dx \\ &= \int_I (\varphi(x)\psi'(x))'f(x)\,dx.\end{aligned} \tag{3.1}$$

As $C^2(I)$ is dense in $L_p(I)$ and $A_n(f,\psi)$ is uniformly bounded in $L_p(I)$, by Banach-Steinhaus theorem, (3.1) holds in $L_p(I)$.

Observing the functional sequence $\{A_n(f,\psi)\}$, setting

$$h_n(x) = n(T_n(f,x) - f(x))$$

for $p > 1$ as $h_n = O(I)$, for every $\psi \in C_0^2(I)$ by weakly sequentially compat principle, there exists $\{n_j\}$ and $h \in L_p(I)$ satisfying

$$\lim_{j\to\infty} A_{n_j}(f,\psi) = \int_I \psi(x)\,dh(x).$$

Hence for $\psi \in C_0^2(I)$, we get the following integral equation

$$\int_I f(x)(\varphi(x)'\psi_x')'\,dx = \begin{cases} \int_I h(x)\psi(x)\,dx, & p > 1, \\ \int_I \psi(x)dh(x), & p = 1. \end{cases} \tag{3.2}$$

We can easily verify the following function is a special solution of (3.2).

$$f_0(x) = \begin{cases} \int_\varsigma^x \frac{1}{\varphi(t)} \int_0^t h(u)dudt, & p > 1, \\ \int_\varsigma^x \frac{1}{\varphi(t)} \int_0^t dh(u)dt, & p = 1, \end{cases}$$

where $\varsigma \in I^0$.

On the other hand by the proof of Theorem 1, we know $f \equiv constant$ (a.e.) is the general solution of the following homogeneous integral equation

$$\int_I f(x)(\varphi(x)\psi'(x))'\,dx = 0,$$

therefore $f(x) = f_0(x) + constant$ is the general solution of (3.2).

Hence, we have to prove only $f_0(x) \in S_p$. In fact for $p > 1$,

$$\varphi(x)f_0''(x) = -\frac{1+2\alpha x}{\varphi(x)}\int_0^x h(u)du + h(x)$$

using Hardy inequality, we obtain

$$\|\varphi(x)f_0''(x)\|_p \le A_p\|h\|_p + \|h\|_p < +\infty.$$

For $p = 1$,

$$\varphi(x)f_0'(x) = \int_0^x dh(x),$$

which implies $\|\varphi(x)f_0'(x)\|_{BV} \le \|h\|_{BV} < +\infty$. Therefore the theorem is proved.

By Lemma 9 and Theorem 2, we obtain the following $L_p$−saturation theorem

**Theorem 3** *Let* $f \in L_p(I), p > 1$. *Then*

$$\| T_n f - f \|_p = O(\frac{1}{n})$$

*iff* $f \in S_p$.

## References


[1] Chen Wenzhong, Approx. Theory & Its Appl., **5:2**(1989), 9–24.
[2] M.O. Heilmen, Approx. Theory & Its Appl., **5:1**(1989), 105–127.
[3] M.O. Heilmen, J. Approx. Theory, **54**(1988), 260–273.
[4] M.M. Derriennic, J. Approx. Theory, **31**(1981), 325– 343.
[5] Z. Ditzian & K. Ivarov, J. Approx. Theory, **56**(1989), 72–90.
[6] Chen Wenzhong, *Approximation Theory of Operators*, Xiamen University Press, 1989, (in Chinese).
[7] Z. Ditzian & V. Totik, *Moduli of Smoothness*, Springer- Verlag, Berlin/New York, 1987.
[8] R.A. DeVore, *The Approximation of Continueous Functions by Positive Linear Operator*, Berlin, 1972.


# 混合指数型积分算子的$L_p$饱和性


陈文忠　　崔振录

(厦门大学数学系,361005)


## 摘　　要


本文讨论了混合指合数型积分算子在$L_p$空间的性质,建立了该类算子的$L_p$饱和定理.