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Global Approximation Theorems for Modified Szász Operators in Exponential Weight Spaces *

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Let $M_n(f,x)$ be the well known Szász operators, i.e.

$$M_n(f,x)=\sum_{k=0}^{\infty}e^{-nx}\frac{(nx)^k}{k!}f(\frac{k}{n}).$$

We propose modified Szász operators as follows

$$L_n(f,x)=\sum_{k=0}^{\infty}S_{n,k+1}(x)n\int_0^{\infty}f(t)S_{n,k}(t)dt+f(0)S_{n,0}(x), \tag{1}$$

where

$$S_{n,k}(x)=e^{-nx}(nx)^k/k!.$$

The object of this paper is to study global approximation for operator (1) for continuous functions on $[0,\infty)$ with oexponential growth.

Using some simple calculations one may verify the following

$$\begin{aligned}&L_n(1,x)=1,\\&L_n((t-x),x)=0,\\&L_n((t-x)^2,x)=\frac{2x}{n}.\end{aligned}$$

Let us introduce the usual notations. Supose

$$C_A=\{f\in C(0,\infty),f(x)=O(e^{Ax})\quad x\to\infty\}.$$

If $f\in C_A$ we define that $\|f\|_A=\sup\limits_{x\geq 0}e^{-Ax}|f(x)|$. The corresponding Lipschitz classes are given for $0<\alpha\leq 2$ by $(h>0)$

$$\begin{aligned}&\Delta_h^2f(x)=f(x+2h)-2f(x+h)+f(x),\\&\omega_A^2(f,\delta)=\sup_{0<h\leq\delta}\|\Delta_h^2f\|_A,\\&\mathrm{Lip}_A^2\alpha=\{f\in C_A,\omega_A^2(f,\delta)=O(\delta^\alpha)\quad\delta\to 0+\}.\end{aligned}$$

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In this paper we will give a necessary and sufficient condition on the rate of convergence of $L_n(f,x)$ to $f(x)$ for $f(x) \in \mathrm{Lip}_A^2\alpha, 0 < \alpha < 2$. The main result is given by the following theorem.

Theorem *If $f \in C_A$ then for $0 < \alpha < 2$ the following are equivalent*

(1) $$e^{-Ax}|L_n(f,x) - f(x)| \le M(\frac{x}{n})^{\alpha/2} \text{ for } n \ge 2A + x;$$

(2) $$f \in \mathrm{Lip}_A^2\alpha.$$

As well as in [1] [3] the method of the proof is the elementry one that was introduced by Berens and Lorentz [2]. Throughout, M is used to denote a positive constant that depends on A, but independently of n and x, and it may represent different values at different occurrences.

Lemma 1 *For $x \ge 0, n \ge 2A + x$, we have*

$$L_n(e^{At},x) \le Me^{At},$$
$$L_n(e^{At}(t-x)^2,x) \le Me^{At}\frac{x}{n}.$$

Lemma 2 *If $f \in C_A^2 = \{f \in C_A, f', f'' \in C_A\}$ then for $n \ge 2A + x$ we have*

$$e^{-Ax}|L_n(f,x)| \le M\frac{x}{n}\|f\|_A.$$

Lemma 3 *If $f \in C_A$ for $x \ge 0, n \ge 2A + x$ we have*

$$e^{-Ax}|L''_n(f,x)| \le Mn^2\|f\|_A.$$

Lemma 4 *If $f \in C_A$ for $x > 0$ and $n \ge 2A + x$ we have*

$$e^{-Ax}|L''_n f(x)| \le M\frac{n}{x}\|f''\|_A.$$

Lemma 5 *If $f \in C_A^2$ for $n \ge 2A + x$ we have*

$$e^{-Ax}|L''_n(f,x)| \le M\|f''\|_A.$$

Theorem 1 *If $f \in C_A, x \ge 0$ then there holds for $n \ge 2A + x$*

$$e^{-Ax}|L_n(f,x) - f(x)| \le M\omega_A^2(f,\sqrt{x/n}). \tag{2}$$

In particular, if $f \in \mathrm{Lip}_A^2\alpha$ for some $\alpha \in (0,2]$ then

$$e^{-Ax}|L_n(f,x) - f(x)| \le M(\frac{x}{n})^{\alpha/2}. \tag{3}$$

Proof To prove Theorem 1 we introduce the (modified) Steklow means (cf [1]) for $h > 0$ by

$$f_h(x) = (\frac{2}{h})^2 \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} [2f(x+s+t) - f(x+2(s+t))] dsdt.$$

One has

$$f(x) - f_h(x) = (\frac{2}{h})^2 \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} \Delta_{s+t}^2 f(x) dsdt,$$

$$f_h''(x) = h^{-2}[8\Delta_{\frac{h}{2}}^2 f(x) - \Delta_h^2 f(x),$$

and hence

$$\|f - f_h\|_A \le \omega_A^2(f,h), \quad \|f_h''\|_A \le 9h^{-2}\omega_A^2(f,h). \tag{4}$$

Note that for $x = 0$ the assertion is trivial. For $f \in C_A, h > 0$ one has by Lemma 1, 2 and (4) for $n \ge 2A + x$ that

$$e^{-Ax}|L_n(f,x) - f(x)| \le M\omega_A^2(f,h)[1 + h^{-2}\frac{x}{n}].$$

So that (2) thereby (3) follows upon setting $h = \sqrt{x/n}$.

Theorem 2 *If $f \in C_A$ satisfies for some $\alpha \in (0,2)$ and $x \ge 0, n \ge 2A + x$*

$$e^{-Ax}|L_n(f,x) - f(x)| \le M(\frac{x}{n})^{\alpha/2}, \tag{5}$$

then

$$f \in \mathrm{Lip}_A^2 \alpha. \tag{6}$$

Proof First we have from (5) for $h \le 1, x \ge 0, n \ge 2A + x$

$$e^{-Ax}|f(x) - 2f(x+h) + f(x+2h)|$$

$$\le M(\frac{x+2h}{n})^{\alpha/2} + e^{-Ax} \int_0^h \int_0^h |L_n''(f - f_\delta, x+s+t)| dsdt$$

$$+ e^{-Ax} \int_0^h \int_0^h |L_n''(f_\delta, x+s+t)| dsdt = J_1 + J_2 + J_3. \tag{7}$$

We write

$$J_1 = M(\frac{x+2h}{n})^{-\alpha/2} \le M[\max(\frac{1}{n^2}, \frac{x+2h}{n})]^{\alpha/2}. \tag{8}$$

Using Lemma 5 and (4) for $x \ge 0, n \ge 2A + x$ one has

$$J_3 \le M\frac{h^2}{\delta^2}\omega_A^2(f,\delta). \tag{9}$$

Note that (see [1] Lemma 10)

$$\int_0^h \int_0^h \frac{1}{x+s+t} dsdt \frac{Mh^2}{x+2h},$$

and using Lemma 4 and (4) we have for $x > 0, n \geq 2A + x$

$$J_2 \leq Me^{-Ax}e^{Ax+2h}||f - f_\delta||_A n \int_0^h \int_0^h \frac{1}{x+s+t} dsdt \leq M\frac{n}{x+2h}h^2\omega_A^2(f,\delta). \tag{10}$$

For the case $x = 0$ since the existence of the integrals for $x = 0$ and the continuity of the expressins involved the estimate (10) holds true. Using Lemma 3 for $x \geq 0, n \geq 2A + x$, we have

$$J_2 \leq Mn^2h^2\omega_A^2(f,\delta).$$

Therefore

$$J_2 \leq Mh^2\omega_A^2(f,\delta)\min(n^2, \frac{n}{x+2h}). \tag{11}$$

Let $\delta_{n,x}^2 = \max\{1/n^2, (x+2h)/n\}$. Then $\delta_{n+1,x} > \frac{3}{4}\delta_{n,x}$ for $n \geq 4$ and for every $\delta < \min\{\frac{1}{4A}, \frac{1}{4}\}$ and every x, n can be chosen such that (see [3] p.260)

$$\frac{3}{4}\delta_{n,x} < \delta < \delta_{n,x}, \tag{12}$$

and so $n \geq 2A + x$.

Hence from (7)–(12) we get

$$e^{-Ax}|f(x) - 2f(x+h) + f(x+2h)|M[\delta_{n,x}^\alpha + \omega_A^2(f,\delta)\left(\frac{h^2}{\delta_{n,x}^2} + \frac{h^2}{\delta^2}\right)]$$

$$\leq M[\delta^\alpha + (\frac{h}{\delta})^2\omega_A^2(f,\delta)]. \tag{13}$$

For the proof of Theorem 2, (13) is sufficient (cf. [1] [3]).

References

[1] M. Becker, *Global approximation theorem for Szász- Mirakjan and Baskakov operators in polynomial weight spaces,* Incdiana Univ. Math. J. Vol. 27(1987) 127-142.

[2] H. Berens and G.G. Lorentz, *Inverse theorems for Bernstein polynomials,* Indiana Univ. Math. J. 21(1972) 693-708.

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修正 Szász 算子在指数权空间的整体逼近

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摘　　要

本文给出用修正 Szász 算子的逼近度刻画指数权空间中类 Lip $_A^2\alpha$ 的一个特征.