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Interpolating Multivariate Rational Splines of Speical Forms*

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Abstract. Very few results have been achived on interpolating rational splines since R. Schaback pioneered the study of the subject in early seventies, to say nothing of interpolating multivariate rational splines. Given in this paper are a few kinds of interpolating multivariate rational splines in triangulation and quadrilateral partition.

R. Schaback first suggested and studied the interpolating rational splines of special forms which are complicated to carry out for their solutions are determined by nonlinear equations. Starting from some practical problems, R.H. Wang and S.T. Wu concretely studied a few types of interpolating rational splines consisting of both polynomial parts and rational parts which are convenient for application because their solutions can be obtained from definite linear equations. It is well known that in univariate case, any partition is composed of intervals; while the cells of partition in multivariate case may possess different geometric shapes. Therefore interpolating multivariate rational splines are strongly related to the geometric structure of the partition concerned. This paper aims at constructing a few kinds of multivariate rational splines by means of interpolation in triangulation and quadrilateral partition.

Shown in Fig. 1 is a triangulation $\triangle$ of a given polygonal domain D. $\triangle_i, \triangle_j, \triangle_k, \triangle_l, \cdots$ are the cells contained in D. Suppose the common edge of $\triangle_i$ and $\triangle_j$ lies on straight line $\Gamma_{ij}: \ l_{ij}(x,y) = l_{ji}(x,y) = 0$. Let $\bar{\Gamma}_{ij} = \triangle_i \cap \triangle_j$ and $l_{ij}(x,y) = a_{ij}x + b_{ij}y + c_{ij}$.
Denote by $R(x,y)$ the multivariate piecewise rational function whose restriction on $\triangle_i$ has the following form

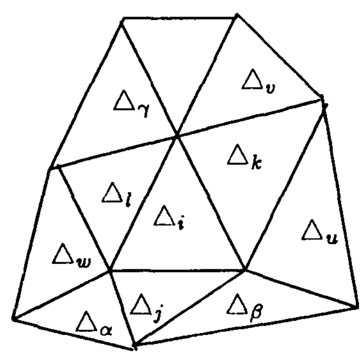

Fig.1

$$\begin{aligned} R(x,y) &= P_i(x,y)/Q_i(x,y) \\ &= (l_{il}(x,y)l_{ik}(x,y)l_{j\alpha}(x,y)l_{j\beta}(x,y))^\mu \sum_{0\le s+t\le \mu-1} c_{st}^{(ij)} x^s y^t \end{aligned}$$

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$$+ \quad (l_{il}(x,y)l_{ij}(x,y)l_{ku}(x,y)l_{kv}(x,y))^{\mu} \sum_{0\le s+t\le \mu-1} c_{st}^{(ik)} x^s y^t$$

$$+ \quad (l_{ij}(x,y)l_{ik}(x,y)l_{lw}(x,y)l_{l\gamma}(x,y))^{\mu} \sum_{0\le s+t\le \mu-1} c_{st}^{(il)} x^s y^t$$

$$+ \frac{(l_{ij}(x,y)l_{ik}(x,y)l_{il}(x,y))^{\mu}}{\left(t_0^{(i)}+t_1^{(i)}x+t_2^{(i)}y\right)^{p}}, \tag{A}$$

where p is an arbitrary positive integer greater than 1, and $c_{st}^{(ij)}, c_{st}^{(ik)}, c_{st}^{(il)}, t_0^{(i)}, t_1^{(i)}, t_2^{(i)}$ are coefficients to be determined.

Definition 1 *A partition is said to be of the type Ω^y if any of its grid-segments is not parallel to the $y-$axis of given coordinate plane while a partition is said to be of the type Ω^x is none of its grid-segments is parallel to the $x-$axis.*

It is always possible for us to choose a proper coordinate system such that the partition is of type Ω^x or Ω^y since the number of grid- segments of a given partition is finite.

Take μ distinct points $t_m^{(ij)}, t_m^{(ik)}$ and $t_m^{(il)}$ on $\bar{\Gamma}_{ij}, \bar{\Gamma}_{ik}$ and $\bar{\Gamma}_{il}$ which are different from the three vertices of $\triangle_i$ respectively. Suppose $t_m^{(ij)} = (x_m^{(ij)}, y_m^{(ij)}), m = 0, 1, \cdots, \mu-1$.

For a partition $\triangle$ of type Ω^y, we give the following interpolation conditions on $\triangle_i$

$$R_y^{(n)}(t_m^{(ij)}) = f_{m,n}^{(ij)}, \quad m = 0,1,\cdots,\mu-1-n; n = 0,1,\cdots,\mu-1; \tag{1}$$

$$R_y^{(n)}(t_m^{(ik)}) = f_{m,n}^{(ik)}, \quad m = 0,1,\cdots,\mu-1-n; n = 0,1,\cdots,\mu-1; \tag{2}$$

$$R_y^{(n)}(t_m^{(il)}) = f_{m,n}^{(il)}, \quad m = 0,1,\cdots,\mu-1-n; n = 0,1,\cdots,\mu-1; \tag{3}$$

$$R_y^{(\mu)}(t_0^{(ij)}) = f_{0,\mu}^{(ij)}; \tag{4}$$

$$R_y^{(\mu)}(t_0^{(ik)}) = f_{0,\mu}^{(ik)}; \tag{5}$$

$$R_y^{(\mu)}(t_0^{(il)}) = f_{0,\mu}^{(il)}, \tag{6}$$

where $R_y^{(n)}(t_m^{(ij)}) = \frac{\partial R(x,y)}{\partial y^n}\big|_{(x,y)=t_m^{(ij)}}$.

Theorem 1 *In a partition of type Ω^y, the bivariate piecewise rational function of the form (A) satisfying the interpolation conditions (1)-(6) turns out to be a uniquely soluble bivariate rational spline belonging to $C^{\mu-1}(D)$.*

Proof We first point out that $R(x,y)$ belongs to the space $C^{\mu-1}(D)$. Without loss of generaity, it suffices to show $R(x,y)$ is in $C^{\mu-1}(\triangle_i \cup \triangle_j)$ for any two neighbouring cells $\triangle_i$ and $\triangle_j$ in D. From the constructure (A) we know $R(x,y)$ on $\triangle_j$ takes the following form

$$\begin{aligned} R(x,y) &= P_j(x,y)/Q_i(x,y) \\ &= (l_{il}(x,y)l_{ik}(x,y)l_{j\alpha}(x,y)l_{j\beta}(x,y))^{\mu} \sum_{0\le s+t\le \mu-1} c_{st}^{(ji)} x^s y^t \\ &+ (l_{ij}(x,y)l_{j\beta}(x,y)l_{\alpha w}(x,y)l_{\alpha b}(x,y))^{\mu} \sum_{0\le s+t\le \mu-1} c_{st}^{(j\alpha)} x^s y^t \end{aligned}$$

$$+ \quad (l_{ij}(x,y)l_{j\alpha}(x,y)l_{\beta u}(x,y)l_{\beta b}(x,y))^{\mu} \sum_{0\le s+t\le \mu-1} c_{st}^{(j\beta)} x^s y^t$$
$$+ \quad \frac{(l_{ij}(x,y)l_{j\alpha}(x,y)l_{j\beta}(x,y))^{\mu}}{\left(t_0^{(j)}+t_1^{(j)}x+t_2^{(j)}y\right)^{p}},$$

where $l_{\alpha b}(x,y)$ and $l_{\beta b}(x,y)$ indicate the outside boundaries of Δ_α and Δ_β (here by boundary $l(x,y)$ we mean that $l(x,y)$ is the linear form of the boundary). Therefore

$$\begin{aligned}&P_i(x,y)/Q_i(x,y)-P_j(x,y)/Q_j(x,y)\\&=(l_{il}(x,y)l_{ik}(x,y)l_{j\alpha}(x,y)l_{j\beta}(x,y))^{\mu}\sum_{0\le s+t\le\mu-1}\left(c_{st}^{(ij)}-c_{st}^{(ji)}\right)x^s y^t+O\left(l_{ij}(x,y)\right)^{\mu}\end{aligned} \quad (7)$$

It is natural to write

$$\sum_{0\le s+t\le\mu-1}\left(c_{st}^{(ij)}-c_{st}^{(ji)}\right)x^s y^t=p_0^{(ij)}(x)+p_1^{(ij)}(x)l_{ij}(x,y)+\cdots+p_{\mu-1}^{(ij)}(x)l_{ij}^{\mu-1}(x,y),$$

where $p_n^{(ij)}(x)$ is a polynomial of degree $\mu-n-1, n=0,1,\cdots,\mu-1$.

We get from (1)

$$\frac{\partial^n}{\partial y^n}\left(\frac{P_i(x,y)}{Q_i(x,y)}-\frac{P_j(x,y)}{Q_j(x,y)}\right)\Big|_{(x,y)=t_m^{(ij)}}=0, m=0,1,\cdots,\mu-n-1; n=0,1,\cdots,\mu-1. \quad (8)$$

Let

$$L_{lk\alpha\beta}(x,y)=(l_{il}(x,y)l_{ik}(x,y)l_{j\alpha}(x,y)l_{j\beta}(x,y))^{\mu}.$$

The fact

$$L_{lk\alpha\beta}\left(x_m^{(ij)},y_m^{(ij)})p_0^{(ij)}(x_m^{(ij)}\right)=0, m=0,1,\cdots,\mu-1$$

and

$$L_{lk\alpha\beta}(x_m^{(ij)},y_m^{(ij)})\neq 0, m=0,1,\cdots,\mu-1$$

imply

$$\frac{\partial}{\partial y}\left(\frac{P_i(x,y)}{Q_i(x,y)}-\frac{P_j(x,y)}{Q_j(x,y)}\right)\Big|_{(x,y)=t_m^{(ij)}}=b_{ij}p_1^{(ij)}(x_m^{(ij)})L_{lk\alpha\beta}(x_m^{(ij)},y_m^{(ij)}).$$

Therefore from (8) we have $p_1^{(ij)}(x_m^{(ij)})=0, m=0,1,\cdots,\mu-2$, i.e., $p_1^{(ij)}(x)\equiv 0$. Similarly we can also prove $p_2^{(ij)}(x)\equiv 0,\cdots,p_{\mu-1}^{(ij)}(x)\equiv 0$. Hence $\frac{P_i(x,y)}{Q_i(x,y)}-\frac{P_j(x,y)}{Q_j(x,y)}=O\left[l_{ij}(x,y)\right]^{\mu}$ which verifies $R(x,y)\in C^{\mu-1}(\Delta_i\cup\Delta_j)$.

We next prove (A) satisfying the conditions (1)–(6) is uniquely soluble. Since

$$\begin{aligned}\frac{P_i(x,y)}{Q_i(x,y)}&=L_{lk\alpha\beta}(x,y)\sum_{0\le s+t\le\mu-1}c_{st}^{(ij)}x^s y^t+O\left[l_{ij}(x,y)\right]^{\mu}\\&=L_{lk\alpha\beta}(x,y)\left[\bar p_0^{(ij)}(x)+\bar p_1^{(ij)}(x)l_{ij}(x,y)+\cdots+\bar p_{\mu-1}^{(ij)}l_{ij}^{\mu-1}(x,y)\right]+O\left[l_{ij}(x,y)\right]^{\mu},\end{aligned}$$

where $\bar{p}_n^{(ij)}(x)$ is a polynomial of degree $\mu - n - 1$. From (1) yield

$$p_0^{(ij)}(x_m^{(ij)}) = f_{m,0}^{(ij)}/L_{lk\alpha\beta}(x_m^{(ij)}, y_m^{(ij)}), \ m = 0, 1, \cdots, \mu - 1,$$

therefore $\bar{p}_0^{(ij)}(x)$ is uniquely soluble. Assume $\bar{p}_1^{(ij)}(x), \cdots, \bar{p}_k^{(ij)}(x)$ are uniquely soluble, then it follows from (1)

$$\bar{p}_{k+1}^{(ij)}(x_m^{(ij)}) = \frac{f_{m,k+1}^{(ij)} - \frac{\partial^{k+1}}{\partial y^{k+1}}\left[L_{lk\alpha\beta}(x,y)\sum_{t=0}^{k}\bar{p}_t^{(ij)}(x)l_{ij}^t(x,y)\right]|_{(x,y)=t_m^{(ij)}}}{(k+1)!b_{ij}^{k+1}L_{lk\alpha\beta}(x_m^{(ij)}, y_m^{(ij)})},$$

where $m = 0, 1, \cdots, \mu - k - 2$. Which means $\bar{p}_{k+1}^{(ij)}(x)$ is uniquely soluble. By induction, the conclusion can be drawn that $\bar{p}_n^{(ij)}(x)$, $n = 0, 1, \cdots, \mu - 1$, hence $c_{st}^{(ij)}, 0 \le s + t \le \mu - 1$, are uniquely soluble. The unique solubility of $c_{st}^{(ik)}$ and $c_{st}^{(il)}$, $0 \le s + t \le \mu - 1$ can be similarly established by means of (2) and (3). Finally we know from (4)–(6) that $t_0^{(i)}, t_1^{(i)}$ are uniquely soluble, which completes the proof of Theorem 1.

For a partition $\triangle$ of type Ω^x, given the following interpolation conditions on $\triangle_i$

$$R_x^{(n)}(t_m^{(ij)}) = g_{m,n}^{(ij)}, \ m = 0, 1, \cdots, \mu - n - 1; n = 0, 1, \cdots, \mu - 1; \tag{9}$$

$$R_x^{(n)}(t_m^{(ik)}) = g_{m,n}^{(ik)}, \ m = 0, 1, \cdots, \mu - n - 1; n = 0, 1, \cdots, \mu - 1; \tag{10}$$

$$R_x^{(n)}(t_m^{(il)}) = g_{m,n}^{(il)}, \ m = 0, 1, \cdots, \mu - n - 1; n = 0, 1, \cdots, \mu - 1; \tag{11}$$

$$R_x^{(\mu)}(t_0^{(ij)}) = g_{0,\mu}^{(ij)}, \tag{12}$$

$$R_x^{\mu}(t_0^{(ik)}) = g_{0,\mu}^{(ik)}, \tag{13}$$

$$R_x^{\mu}(t_0^{(il)}) = g_{0,\mu}^{(il)}, \tag{14}$$

where $R_x^{(n)}(t_m^{(ij)}) = \frac{\partial^n}{\partial x^n}R(x,y)\ |_{(x,y)=t_m^{(ij)}}$.

Theorem 2 *In a partition of type Ω^x, the bivariate piecewise rational function of the form (A) satsifying the interpolation conditions (9)–(14) turns out to be a uniquely soluble bivariate rational spline belonging to $C^{\mu-1}(D)$.*

The proof of Theorem 2 is completely similar to that of Theorem 1.

Let a polygon D on the plane be partitioned in such a way that each cell of D is quadrilateral (it is clear to require that the vertices of every quadrilateral lie neither in the interior nor on the boundary of any other quadrilateral). For the sake of simplicity and also without loss of generality, we suppose D is a rectangular domain and the partition is uniform. Shown in Fig.2 is a retangular partition $\square$ of D. Suppose the common edge of $\square_i$ and $\square_j$ lies on straight line $\Gamma_{ij} : l_{ij}(x,y) = l_{ji}(x,y) = 0$. Let $\bar{\Gamma}_{ij} = \square_i \cap \square_j$.

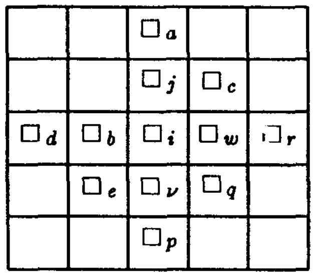

Fig.2

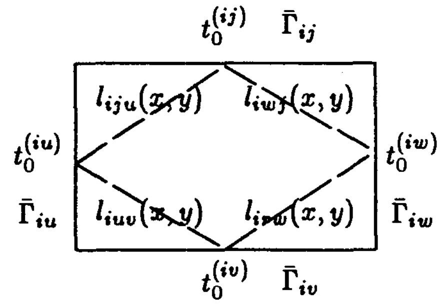

Fig.3

Denote by $R(x,y)$ the multivariate piecewise rational function whose restriction on $\square_i$ has the following form

$$
\begin{aligned}
R(x,y) &= P_i(x,y)/Q_i(x,y)\\
&= (l_{ja}(x,y)l_{jb}(x,y)l_{jc}(x,y)l_{iu}(x,y)l_{iv}(x,y)l_{iw}(x,y))^{\mu}\sum_{0\le s+t\le\mu-1} c_{st}^{(ij)}x^s y^t\\
&+ (l_{ub}(x,y)l_{ud}(x,y)l_{ue}(x,y)l_{iv}(x,y)l_{iw}(x,y)l_{ij}(x,y))^{\mu}\sum_{0\le s+t\le\mu-1} c_{st}^{(iu)}x^s y^t\\
&+ (l_{ve}(x,y)l_{vp}(x,y)l_{vq}(x,y)l_{iw}(x,y)l_{ij}(x,y)l_{iu}(x,y))^{\mu}\sum_{0\le s+t\le\mu-1} c_{st}^{(iv)}x^s y^t\\
&+ (l_{wq}(x,y)l_{wr}(x,y)l_{wc}(x,y)l_{ij}(x,y)l_{iu}(x,y)l_{iv}(x,y))^{\mu}\sum_{0\le s+t\le\mu-1} c_{st}^{(iw)}x^s y^t\\
&+ [(l_{ij}(x,y)l_{iu}(x,y)l_{iv}(x,y)l_{iw}(x,y))^{\mu}] \Big/ \Big[\Big(t_1^{(i)}l_{iju}(x,y)l_{iuv}(x,y) + t_2^{(i)}l_{iuv}(x,y)l_{ivw}(x,y)\\
&+ t_3^{(i)}l_{ivw}(x,y)l_{iwj}(x,y) + t_4^{(i)}l_{iwj}(x,y)l_{iju}(x,y)\Big)^N\Big],
\end{aligned}
$$

where N is an arbitrary positive integer greater than 1, $c_{st}^{(ij)}, c_{st}^{(iu)}, c_{st}^{(iv)}, c_{st}^{(iw)}, t_1^{(i)}, t_2^{(i)}, t_3^{(i)}$ and $t_4^{(i)}$ are coefficients to be determined, and $l_{iju}(x,y), l_{iuv}(x,y), l_{ivw}(x,y)$ and $l_{iwj}(x,y)$ denote the linear forms of the straight lines passing pairs of points $t_0^{(ij)}$ and $t_0^{(iu)}, t_0^{(iu)}$ and $t_0^{(iv)}, t_0^{(iv)}$ and $t_0^{(iw)}$, and $t_0^{(iw)}$ and $t_0^{(ij)}$ respectively. Here $t_0^{(ij)}, t_0^{(iu)}, t_0^{(iv)}$ and $t_0^{(iw)}$ are the four points given on four edges $\bar{\Gamma}_{ij}, \bar{\Gamma}_{iu}, \bar{\Gamma}_{iv}$ and $\bar{\Gamma}_{iw}$ of rectangular cell $\square_i$ respectively as shown in Fig. 3.

Take μ distinct points $t_m^{(ij)}, t_m^{(iu)}, t_m^{(iv)}$ and $t_m^{(iw)}$, $m = 0, 1, \cdots, \mu-1$ not coincident with the four vertices of cell $\square_i$ on $\bar{\Gamma}_{ij}, \bar{\Gamma}_{iu}, \bar{\Gamma}_{iv}$ and $\bar{\Gamma}_{iw}$ respectively.

For a partition $\square$ of type Ω^y, we give the following interpolation conditions on $\square_i$

$$R_y^{(n)}(t_m^{(ij)}) = f_{m,n}^{(ij)},\ m = 0,1,\cdots,\mu-1-n; n = 0,1,\cdots,\mu-1; \tag{15}$$

$$R_y^{(n)}(t_m^{(iu)}) = f_{m,n}^{(iu)},\ m = 0,1,\cdots,\mu-1-n; n = 0,1,\cdots,\mu-1; \tag{16}$$

$$R_y^{(n)}(t_m^{(iv)}) = f_{m,n}^{(iv)},\ m = 0,1,\cdots,\mu-1-n; n = 0,1,\cdots,\mu-1; \tag{17}$$

$$R_y^{(n)}(t_m^{(iw)}) = f_{m,n}^{(iw)},\ m = 0,1,\cdots,\mu-1-n; n = 0,1,\cdots,\mu-1; \tag{18}$$

$$R_y^{(\mu)}(t_0^{(ij)}) = f_{0,\mu}^{(ij)},\ R_y^{(\mu)}(t_0^{(iu)}) = f_{0,\mu}^{(iu)}, \tag{19}$$

$$R_y^{(\mu)}(t_0^{(iv)}) = f_{0,\mu}^{(iv)},\ R_y^{(\mu)}(t_0^{(iw)}) = f_{0,\mu}^{(iw)}, \tag{20}$$

where $R_y^{(n)}(t_m^{(ij)}) = \frac{\partial^n R(x,y)}{\partial y^n}\big|_{(x,y)=t_m^{(ij)}}$.

For a partition $\square$ of type Ω^x, we give the following interpolation conditions on $\square_i$

$$R_x^{(n)}(t_m^{(ij)}) = g_{m,n}^{(ij)},\ m = 0,1,\cdots,\mu-1-n; n = 0,1,\cdots,\mu-1; \tag{21}$$

$$R_x^{(n)}(t_m^{(iu)}) = g_{m,n}^{(iu)},\ m = 0,1,\cdots,\mu-1-n; n = 0,1,\cdots,\mu-1; \tag{22}$$

$$R_x^{(n)}(t_m^{(iv)}) = g_{m,n}^{(iv)},\ m = 0,1,\cdots,\mu-1-n; n = 0,1,\cdots,\mu-1; \tag{23}$$

$$R_x^{(n)}(t_m^{(iw)}) = g_{m,n}^{(iw)},\ m = 0,1,\cdots,\mu-1-n; n = 0,1,\cdots,\mu-1; \tag{24}$$

$$R_x^{(\mu)}(t_0^{(ij)}) = g_{0,\mu}^{(ij)},\ R_x^{(\mu)}(t_0^{(iu)}) = g_{0,\mu}^{(iu)}, \tag{25}$$

$$R_x^{(\mu)}(t_0^{(iv)}) = g_{0,\mu}^{(iv)},\ R_x^{(\mu)}(t_0^{(iw)}) = g_{0,\mu}^{(iw)}, \tag{26}$$

where $R_x^{(n)}(t_m^{(ij)}) = \frac{\partial^n R(x,y)}{\partial x^n}\big|_{(x,y)=t_m^{(ij)}}$.

Theorem 3 *In a partition of type Ω^y (or type Ω^x), the bivariate piecewise rational function of form (B) satisfying the interpolation conditions (15)-(20) (or correspondingly (21)-(26)) turns out to be a uniquely soluble bivariate rational spline belonging to $C^{\mu-1}(D)$.*

We omit the proof of Theorem 3 for its similarity to that of Theorem 1.

The author is indebted to Prof. R.H. Wang for his guidance.

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特殊形式的多元有理样条插值

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摘　要

有理样条插值问题最早是由 R. Schaback 提出的,由于 R. Schaback 考虑此问题时涉及到了非线性方程组的求解,因而实现起来比较复杂. 后来,王仁宏等研究了几类特殊形式的插值有理样条函数,避开了求解非线性方程的困难. 能否在多元情形下建立类似的结果? 本文对此作出了肯定的回答,并就二元情形的三角剖分和四边形剖分建立了几类特殊形式的插值多元有理样条,构造性地证明了解的存在性和唯一性.