Journal of Mathematical Research & Exposition
Vol.13, No.2, May 1993

On Sums of Exponents of Factoring Integers *

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Let $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_t^{\beta_t}$ be the prime factorization of n. Define $h(n) = \min(\beta_1, \beta_2, \cdots, \beta_t)$ and $H(n) = \max(\beta_1, \beta_2, \cdots, \beta_t)$. For convenience take $h(1) = 1$ and $H(1) = 1$.

P.Erdös suggested that it is likely that $\sum_{i=1}^{n} h(i) = n + c\sqrt{n} + o(\sqrt{n})$, where c is a positive constant.

This conjecture was proved by Ivan Niven [1]. In [1], it is proved that

$$\sum_{i=1}^{n} = n + (\varsigma(3/2)/\varsigma(3))\sqrt{n} + o(\sqrt{n})$$

and

$$\lim_{n\to\infty} 1/n \sum_{i=1}^{n} H(i) = 1 + \sum_{k=2}^{\infty} \{1 - \varsigma^{-1}(k)\},$$

where $\varsigma(k)$ is the Riemann zeta- function.

By making use of the results for the k-full numbers and the k-power free integers (see [2]), we prove the following

Theorem 1 $\sum_{i=1}^{n} h(i) = \sum_{k=1}^{n} b_k n^{1/k} + O(n^{1/6} \exp\{-c_1 (\log n)^{4/7} (\log\log n)^{-3/7}\})$, *where* $b_1 = 1, b_2 = \varsigma(3/2)/\varsigma(3), b_3 = \varsigma(2/3)/\varsigma(2) + \prod_p (1 + p^{-4/3} + p^{-5/3}), b_i (4 \le i \le 6)$ *are constants and* $c_1 > 0$.

Theorem 2 $\sum_{i=1}^{n} H(i) = c_0 n + O(n^{1/2} \exp\{-c_2 (\log n)^{3/5} (\log\log n)^{-1/5}\})$, *where* $c_0 = 1 + \sum_{k=2}^{\infty} \{1 - \varsigma^{-1}(k)\}$, *which is equal to 1.7 approximately, and* $c_2 > 0$.

References

[1] Ivan Niven, *Averages of exponents in factoring integers,* Proc. Amer. Math. Soc., **22**(1969), 356-360.

[2] A.Ivic and P.Shiu, *The distribution of powerful integers*, Illinois J. Math., **26**(1982), 576-590.

*Received Feb. 11, 1991.