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Virtual System Method*

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Abstract In this paper, a new method for estimating lower confidence limits of system reliability is proposed. This method, based on the idea of the virtual system, is named as Virtual System Method. It greatly improves the Lindstrom and Madden method with many advantages, such as level consistency and asymptotical efficiency; and it is widely applicable to series parallel cases and many others as a reasonably accurate and quick approximate procedure.

§1. Introduction

Of all the methods of estimating lower confidence limits on system reliability for series systems, the Lindstrom and Maddens method [1] [2] (say simply, the L-M method) is the simplest. Shen's simulation has shown its simple computation and good result [3]. However, Zheng [4] used large sample theory to study the asymptotic properties and found that the L-M confidence limits are not level consistent. And it is our purpose here to improve the L-M method.

In this paper, we propose a new method, Virtual System Method, to solve the question successfully. The lower cinfidence limits of the optimal virtual system (we call it the optimal V-S cinfidence limits) are not only level consistent, but also asymptotically efficient. Its computation is as simple and good as the L-M method. And it can be widely applied to many cases besides series systems.

§2. Improvement on the L-M method

In this section, we consider the simplest system, the series system, i.e.

Model I A series system structure is one where the system fails if and only if at least one subsystem fails. The system reliability is given by

$$R = \prod_{i=1}^{k} R_i, \quad (i = 1, \ldots, k), \tag{2.1}$$

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where R_i is the reliability of the ith component. For the ith component, the mumber of trial is n_i with s_i successes, $i = 1, \ldots, k$.

We want to estimate the lower confidence limit of the system reliability based on the subsystem data. Lindstrom and Maddens [1] [2] gave the L-M method as following:

Let

$$\left\{\begin{aligned} n^* &= \min_{1\le i\le k} n_i, \\ s^* &= n^* \sum_{i=1}^{k} \frac{s_i}{n_i}, \end{aligned}\right. \tag{2.2}$$

and $\hat{R}_{LM}$ satisfy the following equation:

$$\sum_{x=[s^*]+1}^{[n^*]} \binom{[n^*]}{x} \hat{R}_{LM}^x (1-\hat{R}_{LM})^{[n^*]-x} = \alpha, \tag{2.3}$$

where $[n^*]$ and $[s^*]$ denote the integerpart of n^* and s^*, separately. Then $\hat{R}_{LM}$ is called the L-M confidence limit.

Zheng et al [4] used large sample theory to study the asymptotic properties and gave the definition of level consistency:

Definition 2.1 A lower confidence limit $\underline{R}$ is called to be $1-\alpha$ level consistent if

$$\lim_{\min_{1\le i\le k} n_i \to \infty} \Pr\{R \ge \underline{R}\} = 1 - \alpha. \tag{2.4}$$

They found, that the L-M confidence limit $\hat{R}_{LM}$ is not $1-\alpha$ level consistent. So we want to make some improvement on the L-M method. For this purpose, we introduce a new method-Virtual System Method. See Section 3 for definition.

Now review the steps of the L-M method. If we choose n^* and s^* suitably, the solution of the equation (2.3) may by corrected to be level consistent. Here we suggest two alternatives:

(i) Fix $n^* = \min_{1\le i\le k} n_i$, and replace s^* in (2.2) with

$$\tilde{s}^* = n^*(\hat{R} + (\frac{1}{\sqrt{n^*}} - \frac{1}{\sqrt{\hat{n}^{**}}})u_{1-\alpha}\sqrt{\hat{R}(1-\hat{R})}), \tag{2.5}$$

where

$$\hat{n}^{**} = (\frac{1-\hat{R}}{\hat{R}})/(\sum_{i=1}^{k} \frac{1-\hat{R}_i}{n_i \hat{R}_i}),\ \hat{R} = \prod_{i=1}^{k} \hat{R}_i,\ \hat{R}_i = \frac{s_i}{n_i},\ i = 1, \ldots, k,$$

then the solution of the equation (2.3) is $1-\alpha$ level consistent.

However, $\tilde{s}^*$ in (2.5) correlates with the level of confidence $1-\alpha$ and the expression is too complex to evaluate. To overcome these disadvantages, let

(ii)

$$\begin{cases} n^* = \hat{n}^{**} = (\frac{1-\hat{R}}{\hat{R}})/(\sum_{i=1}^{k}\frac{1-\hat{R}_i}{n_i\hat{R}_i}), \\ s^* = [\hat{n}^{**}]\hat{R}, \end{cases} \tag{2.6}$$

where

$$\hat{R} = \prod_{i=1}^{k}\hat{R}_i,\ \ \hat{R}_i = \frac{s_i}{n_i},\ i = 1,\dots,k,$$

then the solution of the equation (2.3) is also $1-\alpha$ level consistent, if we replace n^* and s^* in (2.2) with (2.6).

Remark 1 The proofs can refer to Proposition 3.1 and its remarks.

§3. Virtual System and Virtual System Method

According to the idea of the improvement on the L-M method, we propose Virtual System Method to estimate the lower confidence limit on system reliability. First, we give the definition in general for any kind of systems, for the convenience of later applications.

Definition 3.1 *For any given system with the reliability φ, we arbitarily fix two random variables n^* and s^* depending on subsystem data. When we regard the system with the given subsystem data as a 3-dimensional vector (φ, n^*, s^*), and image a system (only one component) with the reliability φ and s^* successes in n^* trials, then the system of binomial trial corresponding to the 3-dimensional vector (φ, n^*, s^*) is called a virtual system of the given system above. And we write $E^* = (\varphi, n^*, s^*)$.*

Definition 3.2 *For any given system, suppose we obtain a virtual system $E^* = (\varphi, n^*, s^*)$. Given a confidence level $1-\alpha$, let $\hat{R}_{VS}$ satisfy the following equation:*

$$\sum_{x=[s^*]+1}^{[n^*]}\binom{[n^*]}{x}(\hat{R}_{VS})^x(1-\hat{R}_{VS})^{[n^*]-x} = \alpha, \tag{3.1}$$

where $[n^*]$ and $[s^*]$ denote the integerpart of n^* and s^*, separately. Then $\hat{R}_{VS}$ is called the V-S confidence limit of φ depending on n^* and s^* (or on the virtual system $E^* = (\varphi, n^*, s^*)$). And this method (to estimate the lower confidence limit on reliability) is called Virtual System Method.

Proposition 3.1 *Let the system reliability be φ, and the designed confidence level be $1-\alpha$. If n^* and s^* satisfy the condition:*

$$\lim_{n^*\to\infty}\frac{s^*}{n^*} = \varphi, \quad \text{a.s.}, \tag{3.2}$$

then the V-S confidence limit depending on n^* and s^* has the following asymptotical expression:

$$\hat{R}_{VS} = \frac{s^*}{[n^*]} - \frac{u_{1-\alpha}}{\sqrt{n^*}}\sqrt{\varphi(1-\varphi)} + \frac{1}{\sqrt{n^*}}o(1), \quad \text{a.s.,} \tag{3.3}$$

if $n^* \to \infty$, a.s., where $u_{1-\alpha}$ is $1-\alpha$ percentile of standardized normal distribution.

Proof Let $X_1, \cdots, X_{[n^*]}$ be i.i.d. random variables, which satisfy

$$\Pr\{X_1 = 1\} = \hat{R}_{VS}, \quad \Pr\{X_1 = 0\} = 1 - \hat{R}_{VS},$$

and be independent with n^* and s^*. Then the expression (3.1) is equivalent to the following,

$$\Pr\left\{\sum_{i=1}^{[n^*]} X_i > s^*\right\} = \alpha, \tag{3.4}$$

where n^* and s^* are fixed when we calculate the probability. And (3.4) is equivalent to

$$\Pr\left\{\frac{\sum_{i=1}^{[n^*]} X_i - [n^*]\hat{R}_{VS}}{\sqrt{[n^*]\hat{R}_{VS}(1-\hat{R}_{VS})}} > \frac{s^* - [n^*]\hat{R}_{VS}}{\sqrt{[n^*]\hat{R}_{VS}(1-\hat{R}_{VS})}}\right\} = \alpha. \tag{3.5}$$

It follows from Berry-Esseén's theorem that the conditional distribution of the statistic

$$\frac{\sum_{i=1}^{[n^*]} X_i - [n^*]\hat{R}_{VS}}{\sqrt{[n^*]\hat{R}_{VS}(1-\hat{R}_{VS})}}$$

tends to the standardized normal distribution. Then it is not difficult to know that from the expression (3.5) the following holds:

$$\frac{s^* - [n^*]\hat{R}_{VS}}{\sqrt{[n^*]\hat{R}_{VS}(1-\hat{R}_{VS})}} \longrightarrow u_{1-\alpha}, \quad \text{a.s.,} \tag{3.6}$$

if $n^* \to \infty$, a.s..

Noting that $\sqrt{\hat{R}_{VS}(1-\hat{R}_{VS})} \le 1/2$, we know via (3.6) that

$$\lim_{n^* \to \infty}\left(\frac{s^*}{[n^*]} - \hat{R}_{VS}\right) = 0, \quad \text{a.s.,} \tag{3.7}$$

which, combined with condition (3.2), implies that

$$\lim_{n^* \to \infty} \hat{R}_{VS} = \varphi, \quad \text{a.s..} \tag{3.8}$$

By (3.6) and (3.8), we obtain

$$\sqrt{n^*}\left(\hat{R}_{VS} - \frac{s^*}{[n^*]}\right) = -u_{1-\alpha}\sqrt{\varphi(1-\varphi)} + o(1), \quad \text{a.s.,} \tag{3.9}$$

if $n^* \to \infty$, a.s., which concludes the proof.

Remark 2 It follows from (3.9) that

(i) $\sqrt{n^*}(\hat{R}_{VS} - \varphi)$ is asymptotically normal if and only if $\sqrt{n^*}(s^*/[n^*] - \varphi)$ is asymptotically normal;

(ii) $\hat{R}_{VS}$ is $1-\alpha$ level consistent, i.e.,

$$\lim_{n^* \to \infty} \Pr\{\varphi \geq \hat{R}_{VS}\} = 1 - \alpha, \quad \text{a.s.,} \tag{3.10}$$

if and only if $\sqrt{n^*}(s^*/[n^*] - \varphi)/\sqrt{\varphi(1-\varphi)}$ is asymptotically normal with asymptotical distribution $N(0,1)$.

§4. The Optimal Virtual System

In Section 3, we suggest Virtual System Method to improve the LM method to estimate the lower confidence limit on system reliability. But it is an important question to determine a suitable virtual system. Here we propose the optimal virtual system in the sense of level consistency and asymptotical efficiency.

In this section, we shall intrdcuce the optimal virtual system for any kind of systems, based on subsystem test data. And the following three models are our focus:

(1) **Model I:** It is defined in Section 2.

(2) **Model II:** A paraliel system structure is one where the system fails if and only if all its subsystems fail. Its reliability is given by

$$R = 1 - \prod_{i=1}^{k}(1 - R_i), \quad (i = 1, \ldots, k) \tag{4.1}$$

where R_i is the reliability of the ith component. For the ith component, the number of trials is n_i with s_i successes, $i = 1, \ldots, k$.

(3) **Model III:** A series-parallel system is constructed of series or parallel subsystems which are themselves connected in parallel or series. The system reliability φ may be written in terms of the reliabilities of the subsystems using (2.1) and (4.1). Let

$$\varphi = \varphi(R_1, \ldots, R_k), \tag{4.2}$$

where the function φ is known and R_i is the unknown reliability of the ith component. For the ith component, the number of trials is n_i with s_i successes, $i = 1, \ldots, k$.

Now the optimal virtual systems are given below for these three models, separately.

Definition 4.1 *In this paper, we can regard the ith component E_i as same as a 3-dimensional vector (R_i, n_i, s_i), and write*

$$E_i = (R_i, n_i, s_i). \quad (i = 1, \ldots, k) \tag{4.3}$$

Let

(4.4)
$$\begin{cases} R^* = R = \prod_{i=1}^{k} R_i, \\ n^* = \dfrac{1-R}{R} / \sum_{i=1}^{k} \dfrac{1-R_i}{n_i R_i}, \\ s^* = [\hat{n}^*]\hat{R}, \end{cases}$$

where

$$\hat{n}^* = (\frac{1-\hat{R}}{\hat{R}})/(\sum_{i=1}^{k} \frac{1-\hat{R}_i}{n_i \hat{R}_i})$$

is a estimator of n^*, and

$$\hat{R} = \prod_{i=1}^{k} \hat{R}_i, \ \hat{R}_i = \frac{s_i}{n_i}, \ i = 1, \ldots, k.$$

Let

(4.5)
$$E^* = (R^*, \hat{n}^*, s^*).$$

Then the component $E^*(= (R^*, \hat{n}^*, s^*))$ corresponding to the 3- dimensional vector $(R^*, \hat{n}^*, s^*)$ is called the optimal virtual system of the series system. And write

(4.6)
$$E^* = E_1 \ominus E_2 \ominus \cdots \ominus E_k.$$

Definition 4.2 *For Model II, our subsystem components are*

(4.3)
$$E_i = (R_i, n_i, s_i), \quad i = 1, \ldots, k.$$

Let

(4.7)
$$\begin{cases} R^* = R = 1 - \prod_{i=1}^{k}(1 - R_i), \\ n^* = (\dfrac{R}{1-R})/(\sum_{i=1}^{k} \dfrac{R_i}{n_i(1-R_i)}), \\ s^* = [\hat{n}^*]\hat{R}, \end{cases}$$

where

$$\hat{n}^* = (\frac{\hat{R}}{1-\hat{R}})/(\sum_{i=1}^{k} \frac{\hat{R}_i}{n_i(1-\hat{R}_i)})$$

is a estimator of n^*, *and*

$$\hat{R} = 1 - \prod_{i=1}^{k}(1 - \hat{R}_i), \ \hat{R}_i = \frac{s_i}{n_i}, \ i = 1, \ldots, k.$$

Let

$$E^* = (R^*, \hat{n}^*, s^*). \tag{4.8}$$

Then the component E^* defined above is called the optimal virtual system of the parallel system, and write

$$E^* = E_1 \oslash E_2 \oslash \cdots \oslash E_k. \tag{4.9}$$

Definition 4.3 *For Model III, our components are also*

$$E_i = (R_i, n_i, s_i). \quad i = 1, \ldots, k. \tag{4.3}$$

Let

$$\begin{cases} R^* = \varphi = \varphi(R_1, \cdots, R_k), \\ n^* = \varphi(1-\varphi)/(\sum_{i=1}^{k}(\frac{\partial \varphi}{\partial R_i})^2 \frac{R_i(1-R_i)}{n_i}), \\ s^* = [\hat{n}^*]\hat{\varphi}, \end{cases} \tag{4.10}$$

where

$$\hat{n}^* = (\hat{\varphi}(1-\hat{\varphi}))/(\sum_{i=1}^{k}(\hat{\varphi}_i)^2 \frac{\hat{R}_i(1-\hat{R}_i)}{n_i})$$

is a estimator of $n^*, \hat{\varphi} = \varphi(\hat{R}_1, \cdots, \hat{R}_k)$, and

$$\hat{\varphi}_i = \varphi_i(\hat{R}_1, \cdots, \hat{R}_k) = \frac{\partial \varphi(R_1, \cdots, R_k)}{\partial R_i}\bigg|_{R_i = \hat{R}_i}, \quad \hat{R}_i = \frac{s_i}{n_i}, \ i = 1, \ldots, k.$$

Let

$$E^* = (R^*, \hat{n}^*, s^*). \tag{4.11}$$

Then the component E^* is called the optimal virtual system of the series- parallel system.

Remark 3 The trial number n^* of the optimal virtual system in (4.10) may be written in terms of the numbers of the optimal virtual subsystem using (4.4) and (4.7).

Remark 4 The V-S confidence limit depending upon the optimal virtual system is called the optimal V-S confidence limit.

§5. Asymptotic properties of the optimal V-S confidence limits

In this part, we point that the optimal V-S confidece limits are level consistent and asymptotically efficient for any kind of systems. Here we discuss Model III which includes the two others.

For a series-parallel system with system reliability $\varphi = \varphi(R_1, \cdots, R_i)$, assume that its optimal virtual system is

$$E^* = (\varphi, \hat{n}^*, s^*), \tag{5.1}$$

which is defined as Definition 4.3. Then we have the following theorem,

Theorem 5.1 *If the system reliability $\varphi = \varphi(R_1, \cdots, R_k)$ is a differential (or, totally differentiable) function mapping from $(0,1)^k$ to $(0,1)$, i.e.*

$$\varphi : (0,1)^k \longrightarrow (0,1), \tag{5.2}$$

where $(0,1)^k$ denotes a k-dimensional product measurable space $\overbrace{(0,1) \times \cdots \times (0,1)}^{k-fold}$, and the system satisfies the following condition:

$$\min_{1 \le i \le k} |\varphi_i| = \min_{1 \le i \le k} |(\partial \varphi)/(\partial R_i)| > 0. \tag{5.3}$$

Then

(i) The optimal V-S confidence limit $\hat{R}^*_{VS}$ is $1-\alpha$ level consistent, i.e.

$$\lim_{\min_{1 \le i \le k} n_i \to \infty} Pr\{\varphi \ge \hat{R}^*_{VS}\} = 1 - \alpha. \tag{5.4}$$

(ii) The optimal V-S confidence limit $\hat{R}^*_{VS}$ is asymptotically efficient.

Proof

(i) First, it is obvious that

$$\lim_{\min_{1 \le i \le k} n_i \to \infty} n^* = \infty, \tag{5.5}$$

and, from Definition 4.3,

$$\frac{s^*}{[n^*]} = \hat{\varphi}. \tag{5.6}$$

Then, from the propositions and the remarks, we only need to prove that

(a) $\hat{\varphi} \to \varphi$, a.s.,

(b) $\sqrt{n^*}(\hat{\varphi} - \varphi)/(\sqrt{\varphi(1-\varphi)})$ is asymptotically normal with asymptotical distribution $N(0,1)$, when $\min_{1 \le i \le k} n_i \to \infty$.

In fact, it follows from Taylor's theorem

$$\begin{aligned} \hat{\varphi} - \varphi &= \varphi(\hat{R}_1, \cdots, \hat{R}_k) - \varphi(R_1, \cdots, R_k) \\ &= \sum_{i=1}^{k} \varphi_i(\hat{R}_i - R_i) + o(1) \left[\sum_{i=1}^{k} (\hat{R}_i - R_i)^2\right]^{1/2}. \end{aligned} \tag{5.7}$$

From the condition (5.3), it is not difficult to see that

$$n^* \sum_{i=1}^{k} (\hat{R}_i - R_i)^2 = O_p(1). \tag{5.8}$$

So, combining (5.7) and (5.8), we obtain

$$
(5.9)\qquad \begin{aligned}
\sqrt{n^*}(\hat{\varphi}-\varphi) &= \sqrt{n^*}\sum_{i=1}^{k}\varphi_i(\hat{R}_i - R_i) + o(1)\left[n^*\sum_{i=1}^{k}(\hat{R}_i - R_i)^2\right]^{1/2}.\\
&= \sqrt{n^*}\sum_{i=1}^{k}\varphi_i(\hat{R}_i - R_i) + o_p(1)\\
&\to N(0,\varphi(1-\varphi)),
\end{aligned}
$$

when $\min_{1\le i\le k} n_i \to \infty$. Therefore (5.5), (5.7) and (5.9) conclude the proof.

(ii) It follows from the above and the remarks that

$$
(5.10)\qquad \sqrt{n^*}(\hat{R}^*_{LM} - \varphi) \xrightarrow{d} N(-u_{1-\alpha}\sqrt{\varphi(1-\varphi)}, \varphi(1-\varphi)).
$$

Let

$$
R = (R_1,\cdots,R_k)^\tau,\quad \frac{\partial\varphi}{\partial R} = (\frac{\partial\varphi}{\partial R},\cdots,\frac{\partial\varphi}{\partial R_k})^\tau,
$$

and the Fisher's information matrix be $I(R)$, then the Cramér-Rao lower bound of the parameter $\varphi=\varphi(R)$ is given by

$$
(5.11)\qquad \Lambda = \frac{\partial\varphi}{\partial R^\tau} I(R)^{-1}\frac{\partial\varphi}{\partial R}.
$$

Then we only need to prove that

$$
(5.12)\qquad \Lambda = \frac{\partial\varphi}{\partial R^\tau} I(R)^{-1}\frac{\partial\varphi}{\partial R} = \frac{\varphi(1-\varphi)}{n^*}.
$$

In fact, the joint density of the subsystem data is

$$
p = \prod_{i=1}^{k}\binom{n_i}{x_i} R_i^{x_i}(1-R_i)^{n_i - x_i}.
$$

It is easy to see that

$$
I(R) = E(\frac{\partial}{\partial R}\log p\frac{\partial}{\partial R^\tau}\log p) = \mathrm{diag}(\frac{n_1}{R_1(1-R_1)},\cdots,\frac{n_k}{R_k(1-R_k)}),
$$

where $\mathrm{diag}(a_1,\cdots,a_k)$ is a diagonal matrix with the diagonal elements $a_1,\ldots,a_k$ respectively.

Therefore, we obtain

$$
\begin{aligned}
\Lambda &= \frac{\partial\varphi}{\partial R^\tau} I(R)^{-1}\frac{\partial\varphi}{\partial R} = \frac{\partial\varphi}{\partial R^\tau}\mathrm{diag}\left(\frac{R_1(1-R_1)}{n_1},\cdots,\frac{R_k(1-R_k)}{n_k}\right)\frac{\partial\varphi}{\partial R}\\
&= \sum_{i=1}^{k}(\frac{\partial\varphi}{\partial R_i})^2\frac{R_i(1-R_i)}{n_i} = \frac{\varphi(1-\varphi)}{n^*},
\end{aligned}
$$

which completes the proof.

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虚拟系统法

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摘要

本文基于虚拟系统的想法，讨论了设立系统可靠性的置信下限的方法，称为虚拟系统法．该方法克服了Lindstrom 和Madden 方法的保守性．本文提供的方法的优点是，它具有水平相合性，渐近有效性以及计算的简便．